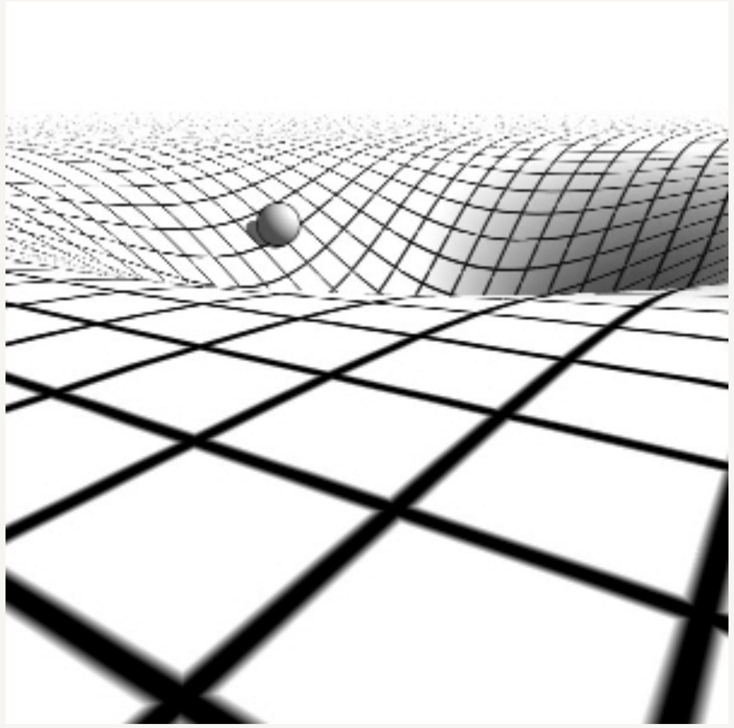
AUTONOMOUS RELATIVISTIC SATELLITE POSITIONING IN PERTURBED SPACE-TIME



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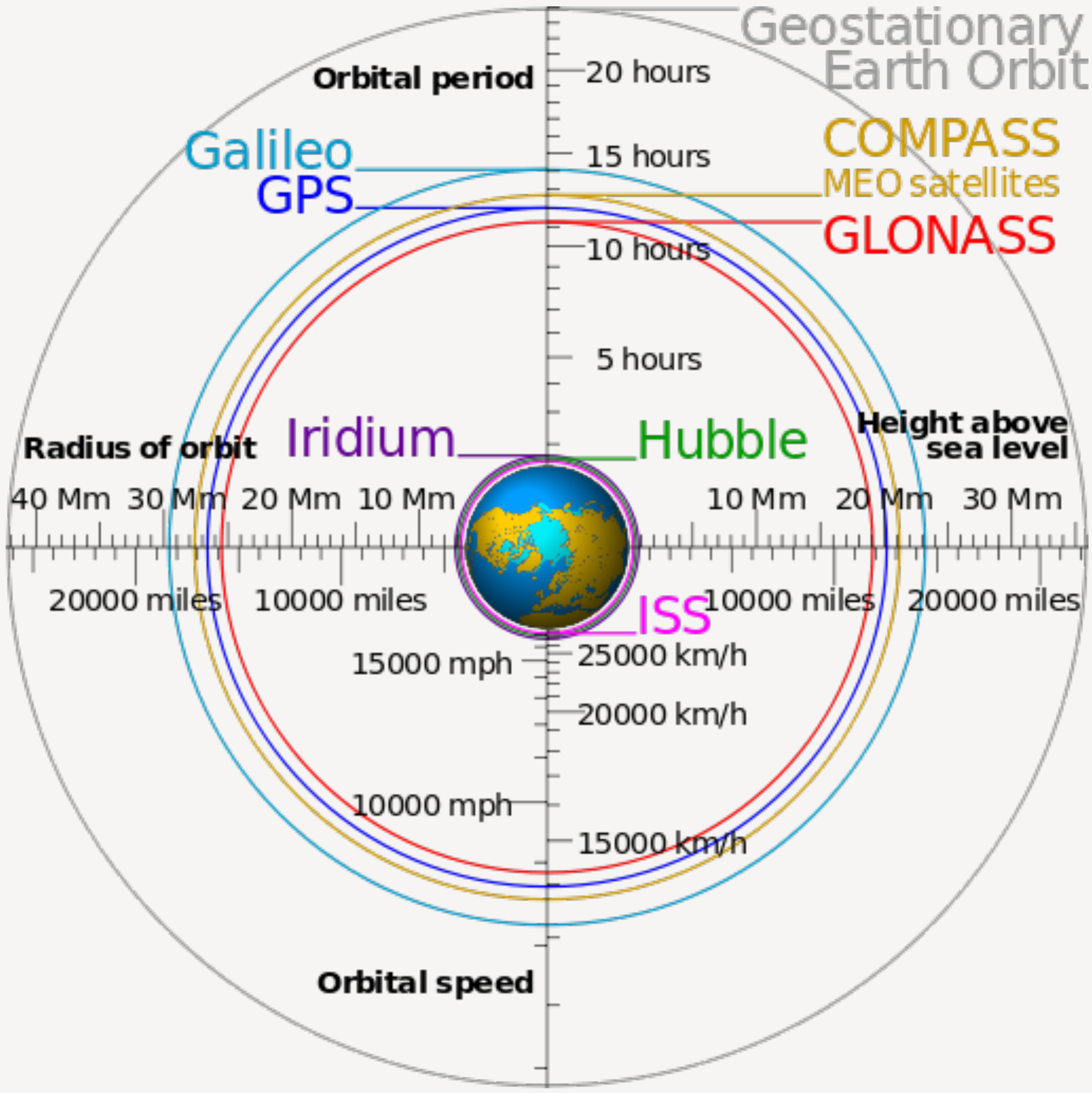
OUTLINE

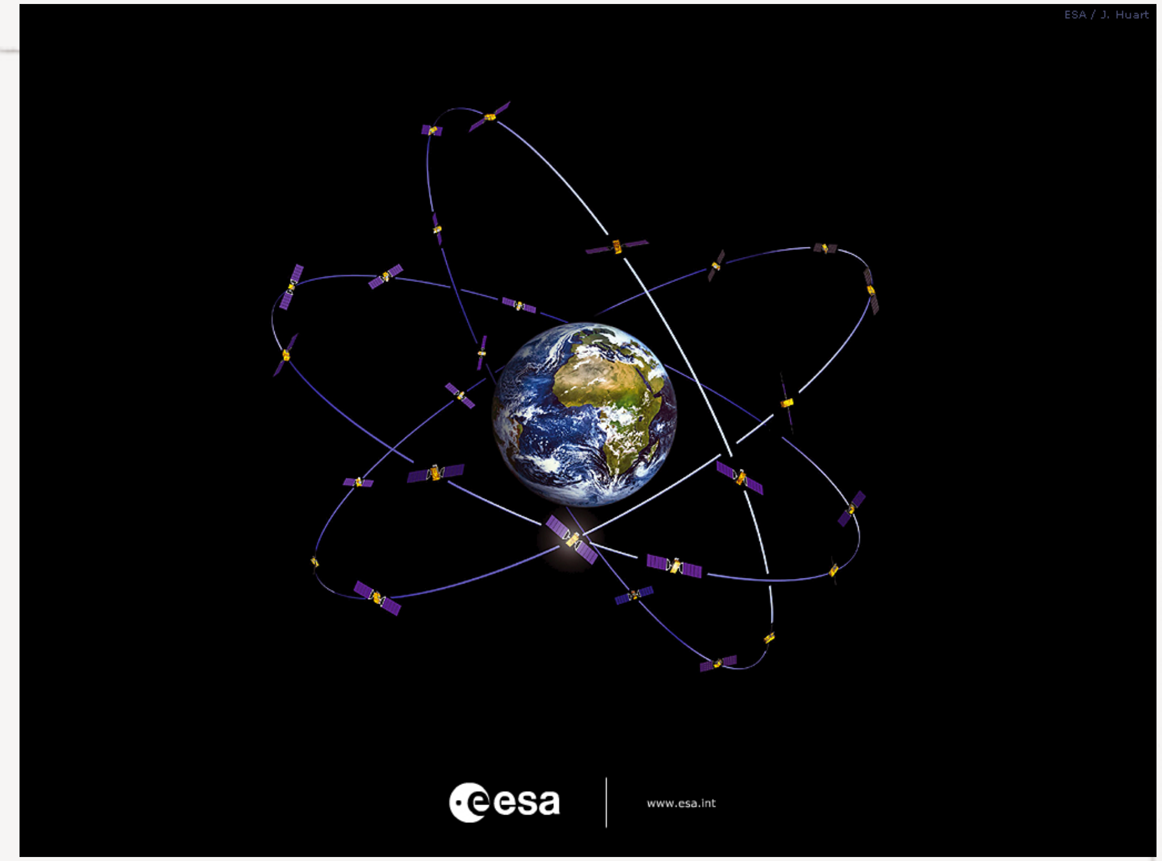
- Motivation: GNSS
- Relativistic Positioning System - RPS (emission coordinates)
- Autonomous Basis of Coordinates concept
- RPS in Schwarzschild space-time
- RPS in “Earth” space-time
- Summary & next steps

GLOBAL NAVIGATION SATELLITE SYSTEMS (GNSS)

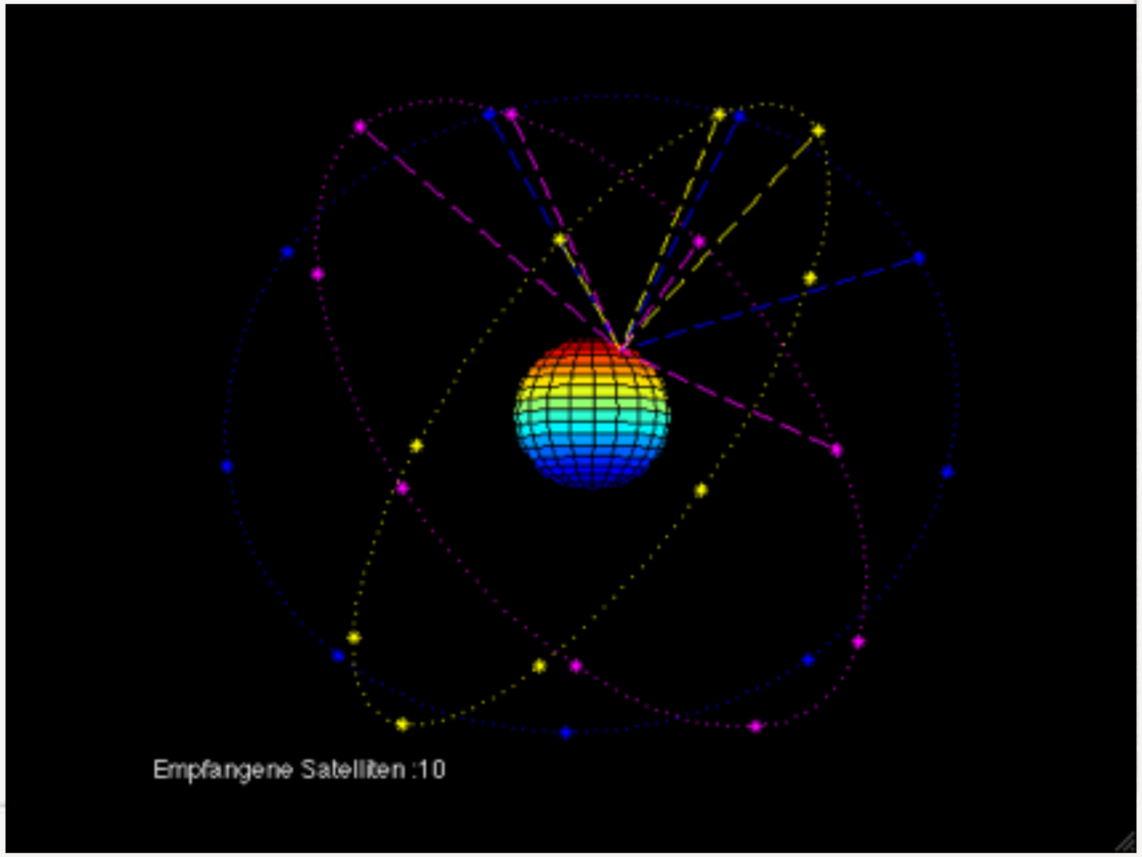
- Global Positioning System (GPS) - USA 
- GLObal NAvigation Satellite System (GLONASS) - Russia 
- GALILEO - Europe 
- BeiDou Navigation satellite System - China 


Geostationary Earth Orbit
Orbital period
20 hours
Galileo
15 hours
COMPASS
MEO satellites
GPS
GLONASS
10 hours
5 hours
Iridium
Hubble
Height above sea level
Radius of orbit
40 Mm
30 Mm
20 Mm
10 Mm
10 Mm
20 Mm
30 Mm
20000 miles
10000 miles
10000 miles
20000 miles
ISS
25000 km/h
15000 mph
20000 km/h
10000 mph
15000 km/h
Orbital speed

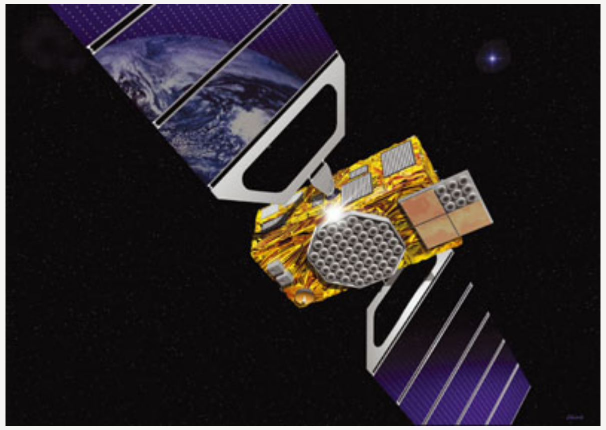
GALILEO



- European Space Agency + EU
- 30 (27 operational + 3 spares) satellites in 3 orbital planes at 56° inclination
- 8 launched
- altitude=23.222 km



GNSS - BASICS



- Atomic clock onboard a satellite sends a signal to a receiver at the time of emission t_E
- Clock in the receiver – receives the signal at time of the reception t_R

- The distance between the receiver and the satellite is: $(t_R - t_E)c$
- We assume that the location of the satellite is known in a given coordinate system
- The receiver is located on the **surface of a sphere** of radius : $(t_R - t_E)c$

GNSS - BASICS
POSITION DETERMINATION

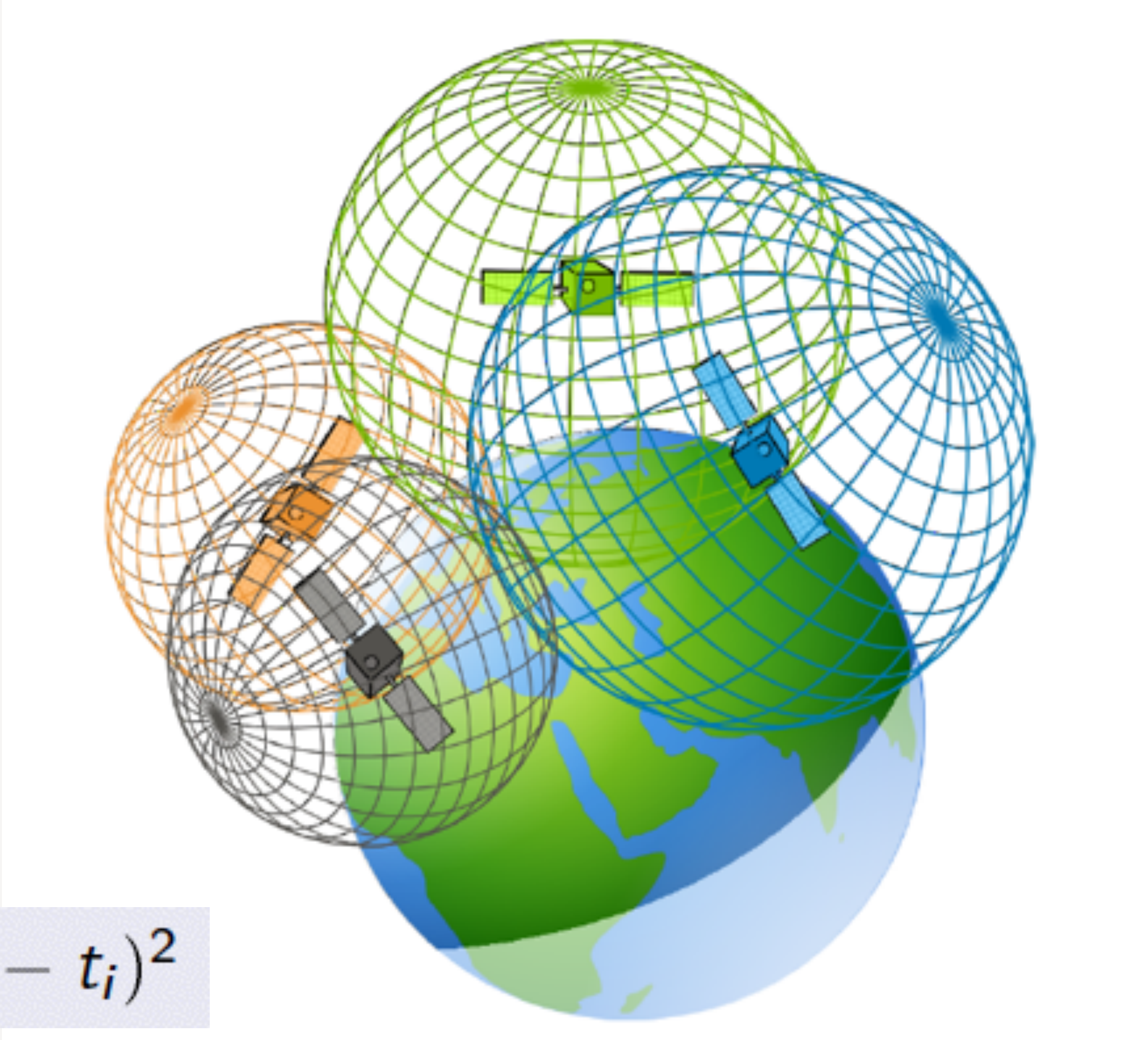
trilateration

i- satellite

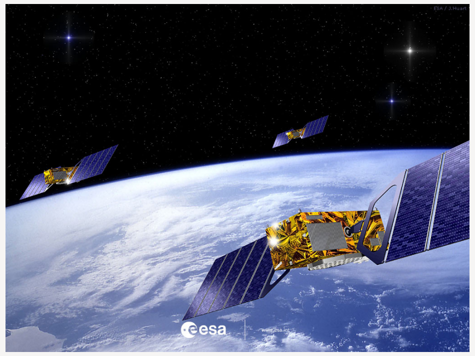
$$S_i = (t_i, x_i, y_i, z_i)$$

equation:

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = c^2(t - t_i)^2$$



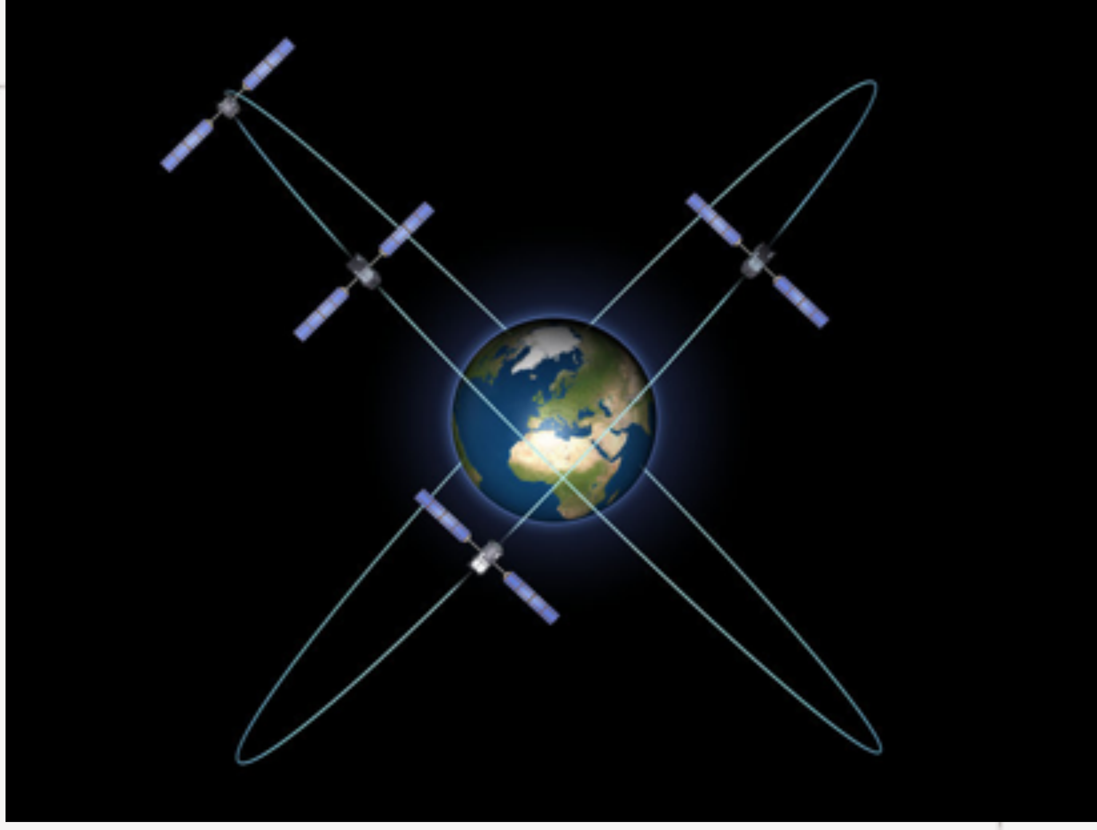
for user's coordinates: x, y, z and t



TIME!

- **accuracy** in time < ns needed to achieve positional accuracy < 30 cm
- **synchronization of clocks** – reference frame!
- **definition of time** - GPS coordinate time - the time of a clock at rest on the geoid - ECEF (Earth Centered Earth Fixed system), Earth Centered Inertial system, International Celestial Reference Frame
- … complicated

PRESENT GNSS



- 4D=3 space + 1 time: 4 satellites
- user receives "clock" information from 4 satellites

  + orbital parameters (determined by ground tracking) :

  → positions of satellites → user's position

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = c^2(t - t_i)^2$$

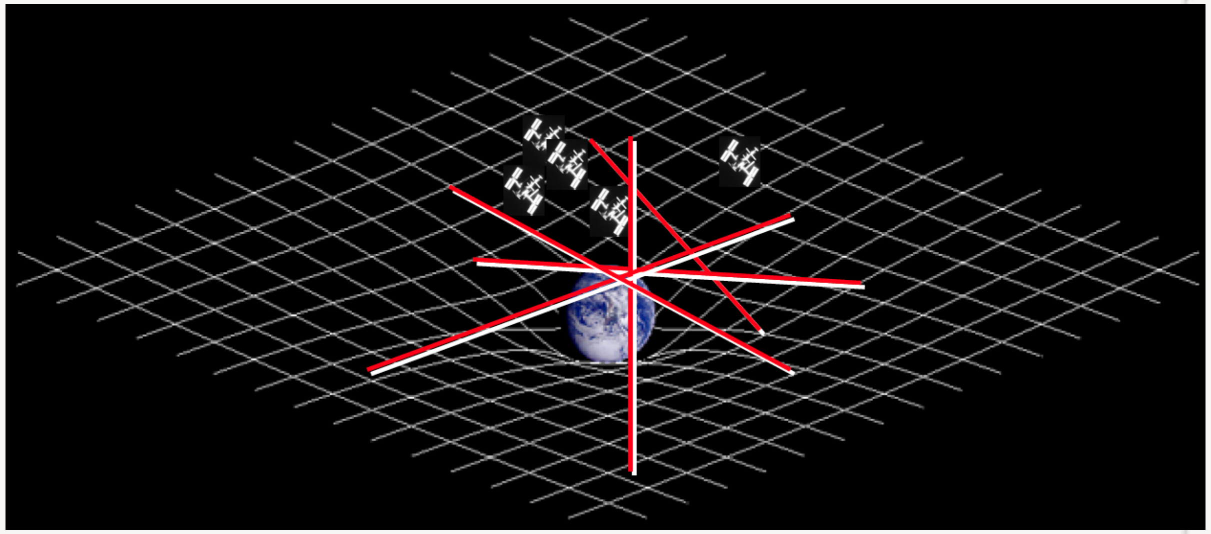
- absolute space and time: reference frame (?) + relativistic corrections

RELATIVITY AND GNSS

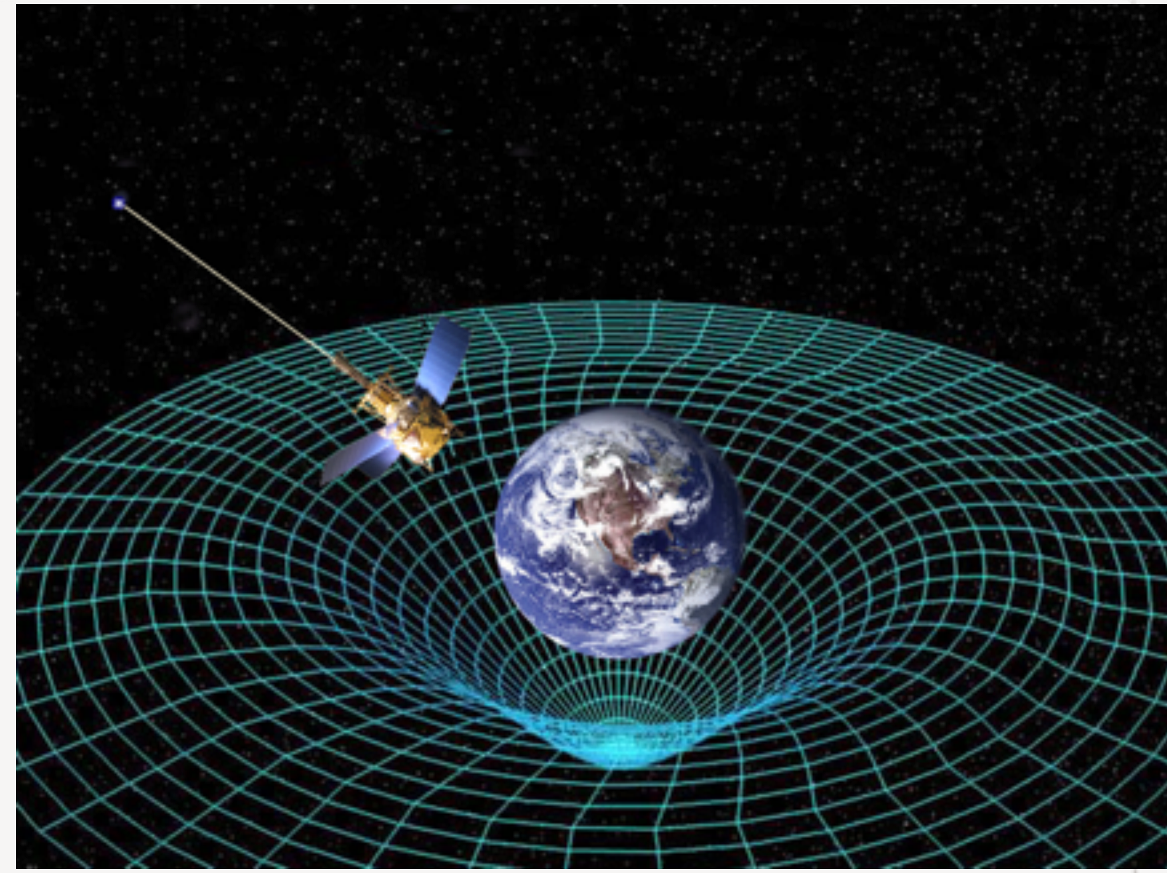
Space and time are not absolute!

GNSS is affected in three ways:

- in the equations of motion of the satellites
- in the signal propagation
- in the **beat rate of the clocks**



RELATIVISTIC CORRECTIONS



clocks on Earth and on a satellite run at different pace:

$$\left(\frac{d\tau_o}{d\tau_s}\right)^2 = \frac{\left(1 - \frac{2GM_\oplus}{r_o c^2}\right) - \frac{v_o^2}{c^2}}{\left(1 - \frac{2GM_\oplus}{r_s c^2}\right) - \frac{v_s^2}{c^2}}$$

- dilatation of time (7.2 μs/day)
- gravitational redshift (45.65 μs/day)
- total: 38.5 μs/day - x=c dt ≈ 12 km/day

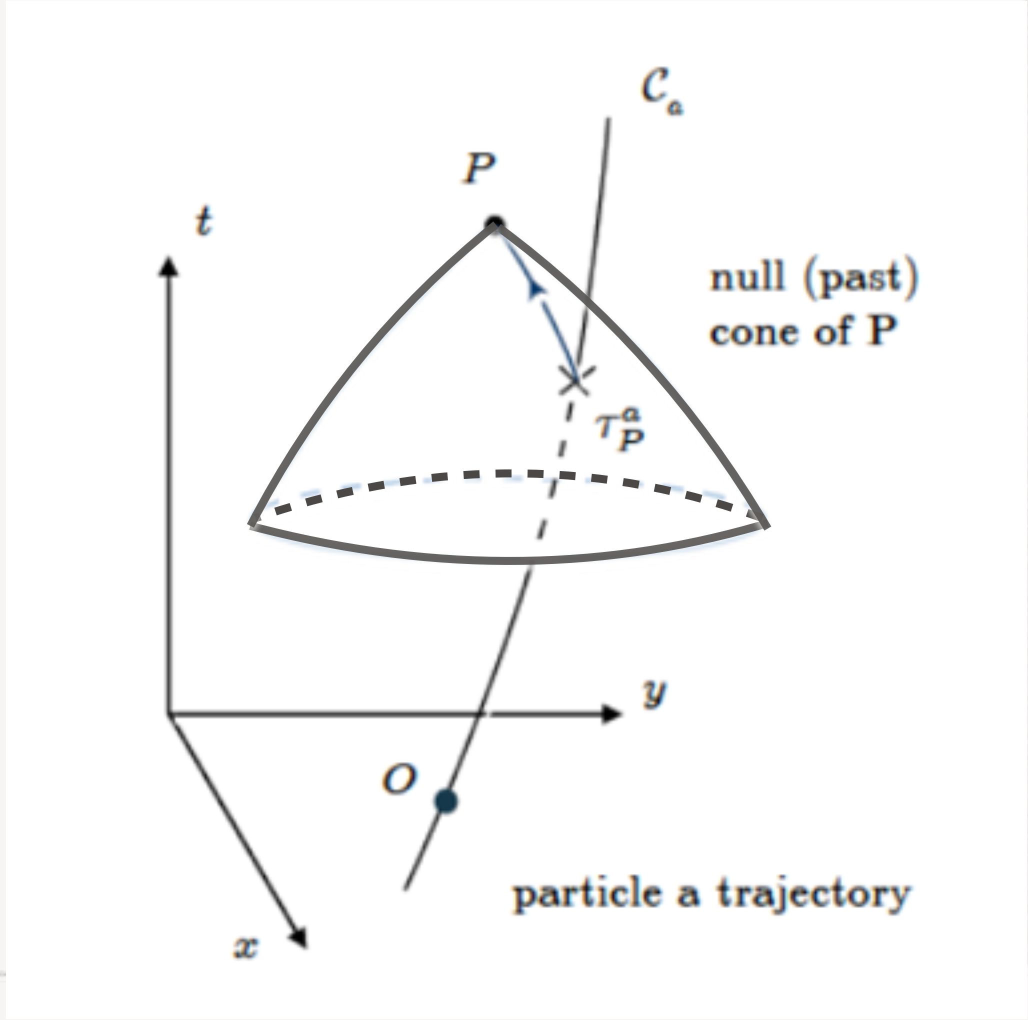
TWO WAYS OF INCLUDING RELATIVITY - CORRECTIONS

- keep absolute time and space and add **corrections** to the level of desired accuracy
- other corrections:
  - quadrupole potential of the Earth
  - Sagnac effect
  - effect of the eccentricity of the orbits, etc. (e.g., Ashby 2003)
  - the Moon and the Sun
- more accurate clocks, more corrections…

ALTERNATIVE APPROACH

- in relativity: space and time are not absolute, proper times of satellites (τ) run at different paces, but each uniquely defines satellite's position along its orbit
- let's have an event P:

past null cone of P crosses the worldline of a satellite C_a at $\tau^P{}_a$

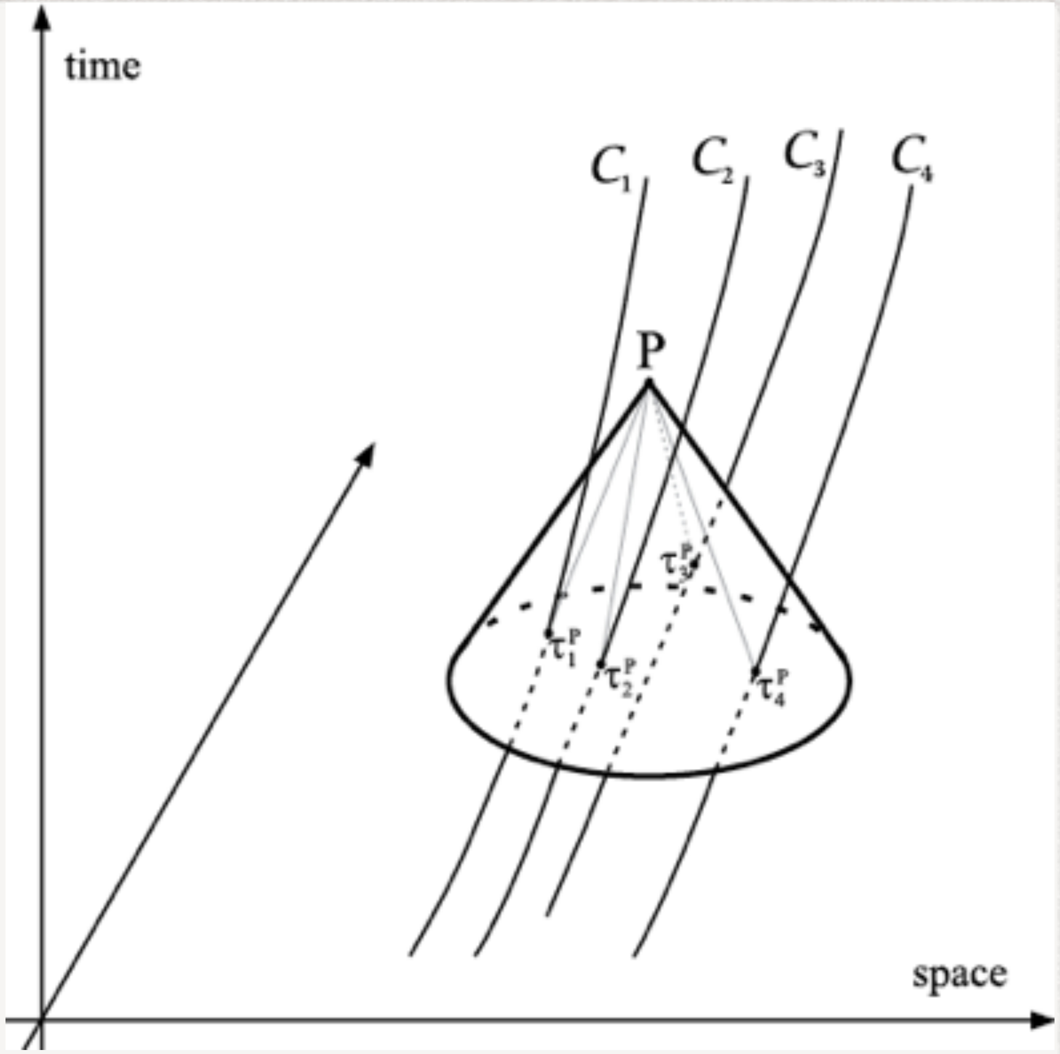


EMISSION COORDINATES

• 4 satellites, 4 worldines:

$(\tau^P{}_1, \tau^P{}_2, \tau^P{}_3, \tau^P{}_4)$ - define event P

• emission coordinates



• instead of (x,y,z,t) use satellites' proper times or emission coordinates $(\tau_1, \tau_2, \tau_3, \tau_4)$

+ orbital parameters (determined by ground tracking):

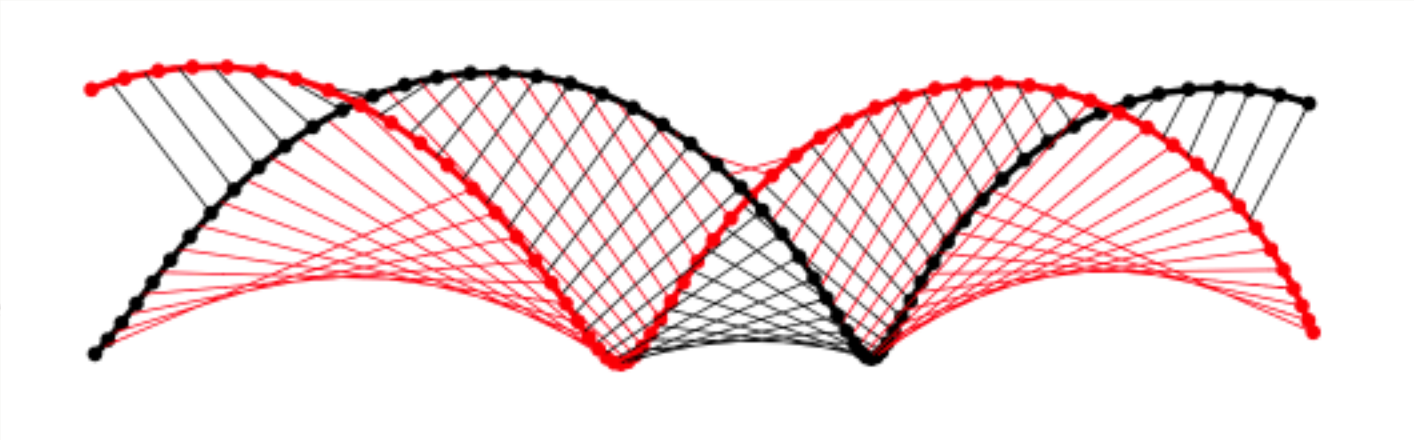
→ positions of satellites → user's position

EMISSION COORDINATES - ADVANTAGES

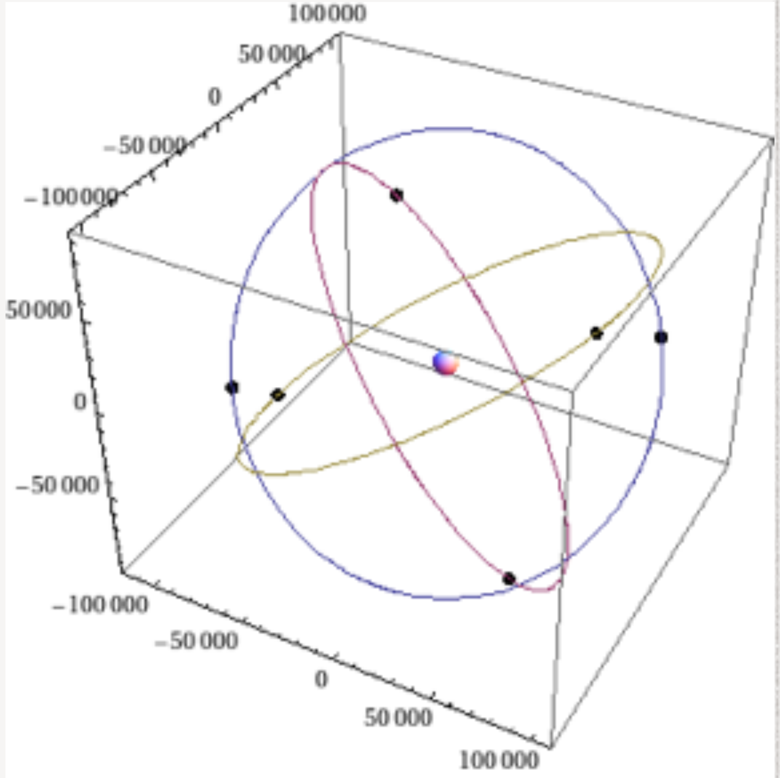
- physical quantities (proper times at the moment of the emission of the signal) measured on-board
- independent of any terrestrial reference frame
- relativistic effects already (naturally) included
- no need to synchronise satellite clocks

ABC CONCEPT

- on-board receivers (inter-satellite links)
- if the user is one of the satellites - from its own and proper times of 3 other satellites it can determine its own position at any given time
- NEXT STEP: let two satellites communicate their proper times to each other for some time
- pairs of (τ_1, τ_2) allow deduction of satellites' own orbital parameters!
- ground tracking not necessary
- Autonomous Basis of Coordinates (ABC)



ARIADNA PROJECTS



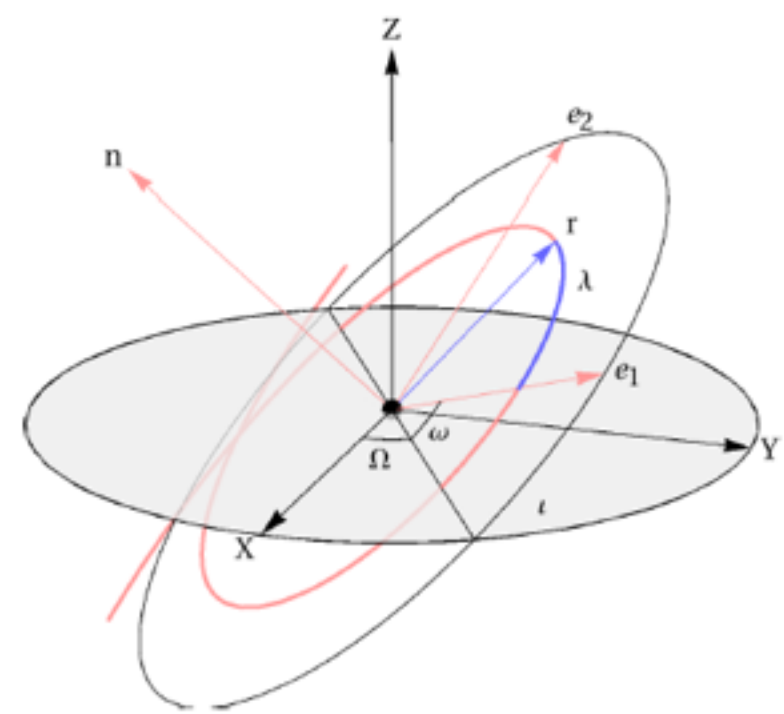
- gravitational field of isolated, spherically symmetric Earth (Schwarzschild metric)
- Ariadna project by Čadež et al. 2010: relativistic positioning system is feasible, stable and accurate

(transformation x,y,z,t ⟷ emission coordinates works)

- Ariadna project by Čadež et al. 2011: inter-satellite communication and orbital parameters determination - ABC concept works

IN SCHWARZSCHILD METRIC

$$g_{\mu\nu} = \begin{bmatrix} -(1-\frac{2GM}{rc^2}) & 0 & 0 & 0 \\ 0 & \frac{1}{1-2GM/rc^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix}$$



- 8 constants of motion: $H,\ a,\ \varepsilon,\ \iota,\ -\tau_a,\ -t_a,\ \omega,\ \Omega$
- analytical solutions for orbits (Kostić 2012)

POSITIONING IN SCHWARZSCHILD SPACE-TIME

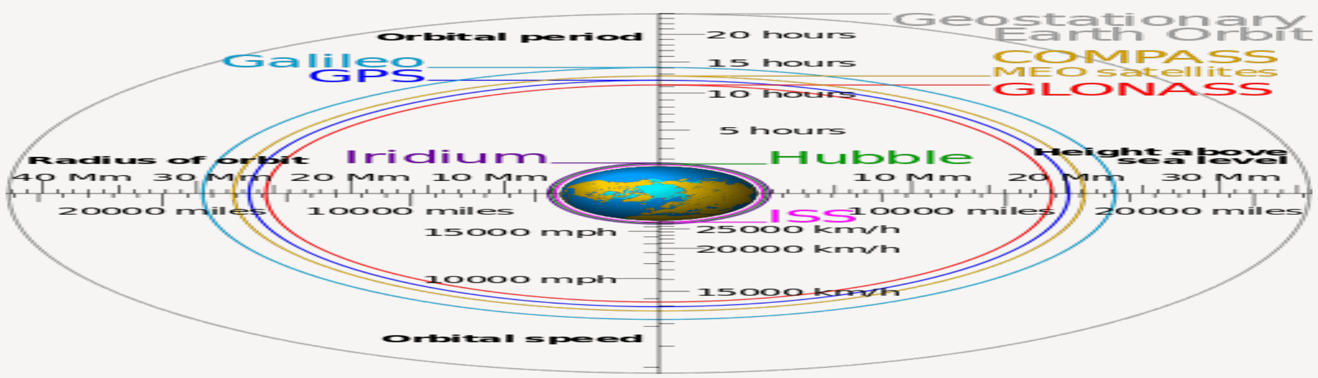
- model positioning: receiver + signals from 4 satellites on known orbits
- exchange of signals: ray-tracing
- proper times $\longrightarrow$ receiver's position
- positioning - **works!**
- relative accuracy 10^{-32} in t, 10^{-27} - 10^{-25} in x,y,z (!)

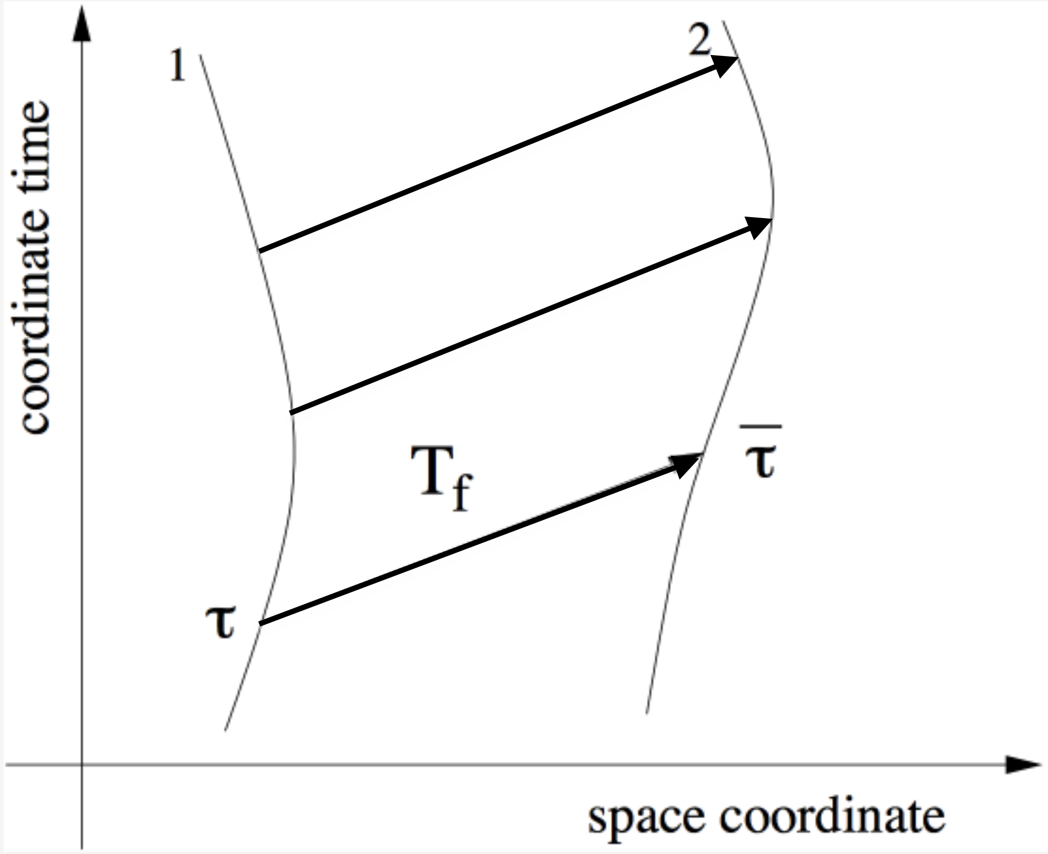
ABC SYSTEM IN SCHWARZSCHILD METRIC

- initially, orbital parameters known only with limited accuracy, use of inter-satellite links (pairs τ_1, τ_2)
- time of flight:

$$T_f(\vec{R}_1, \vec{R}_2) = t_2(\overline{\tau}) - t_1(\tau)$$

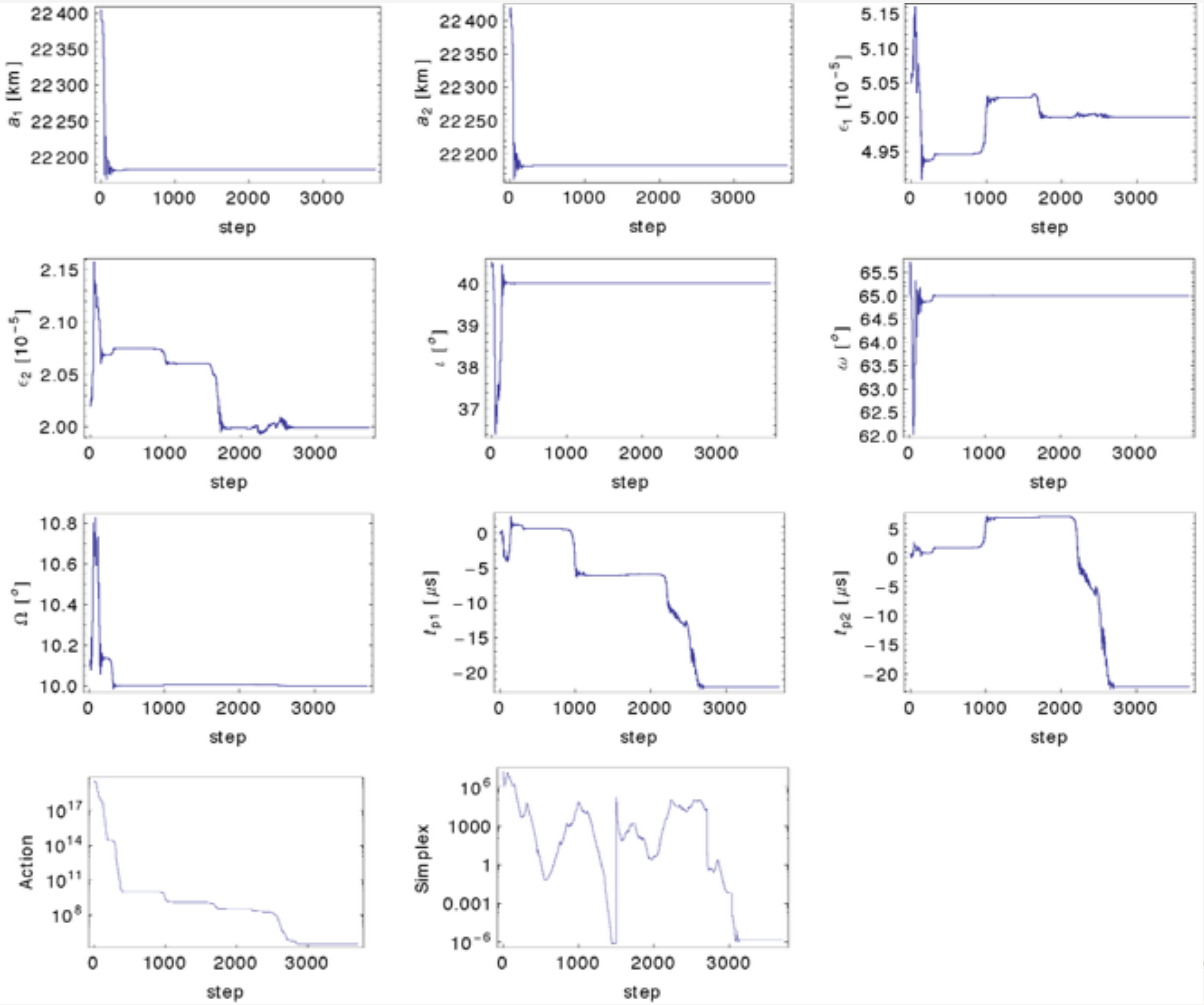
- action S:





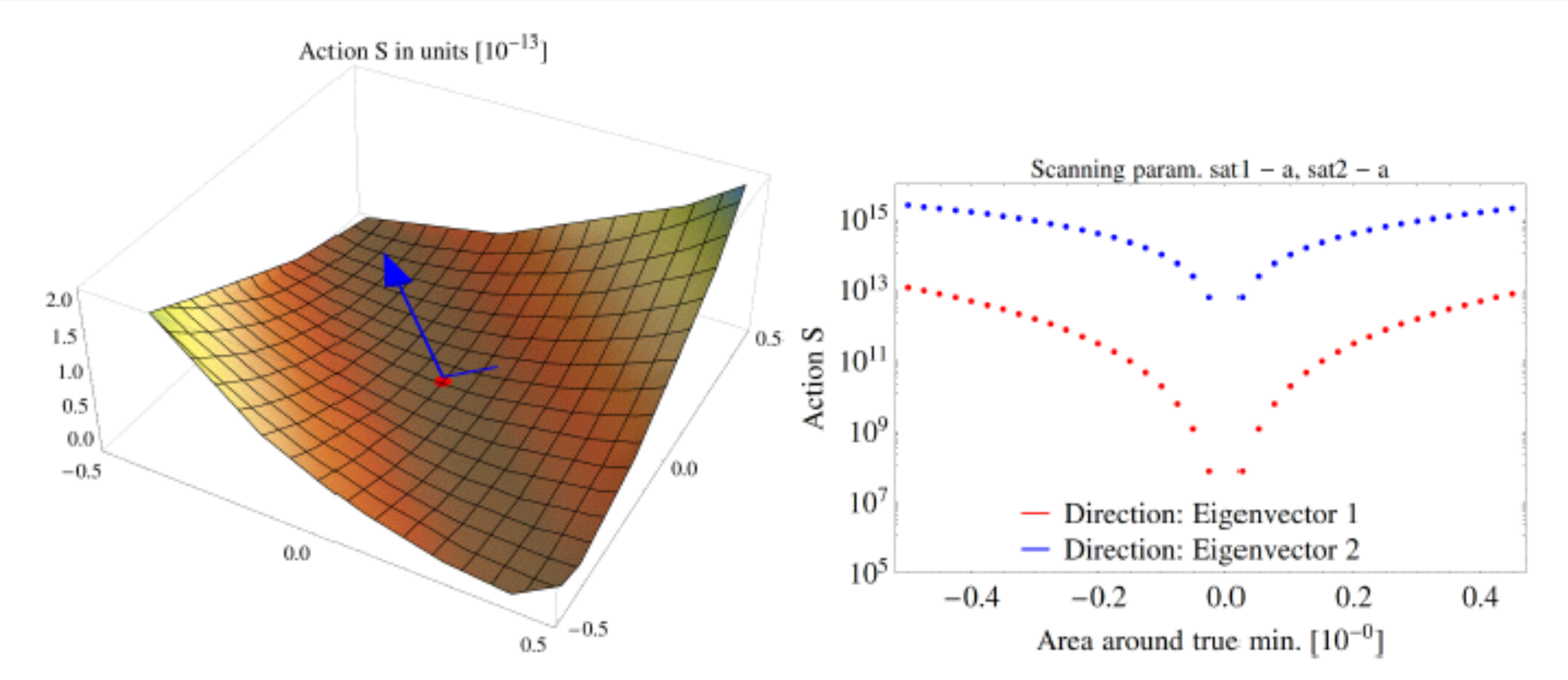
- minimization of S
- possible to refine orbital parameters to relative accuracy 10^{-15} - works!

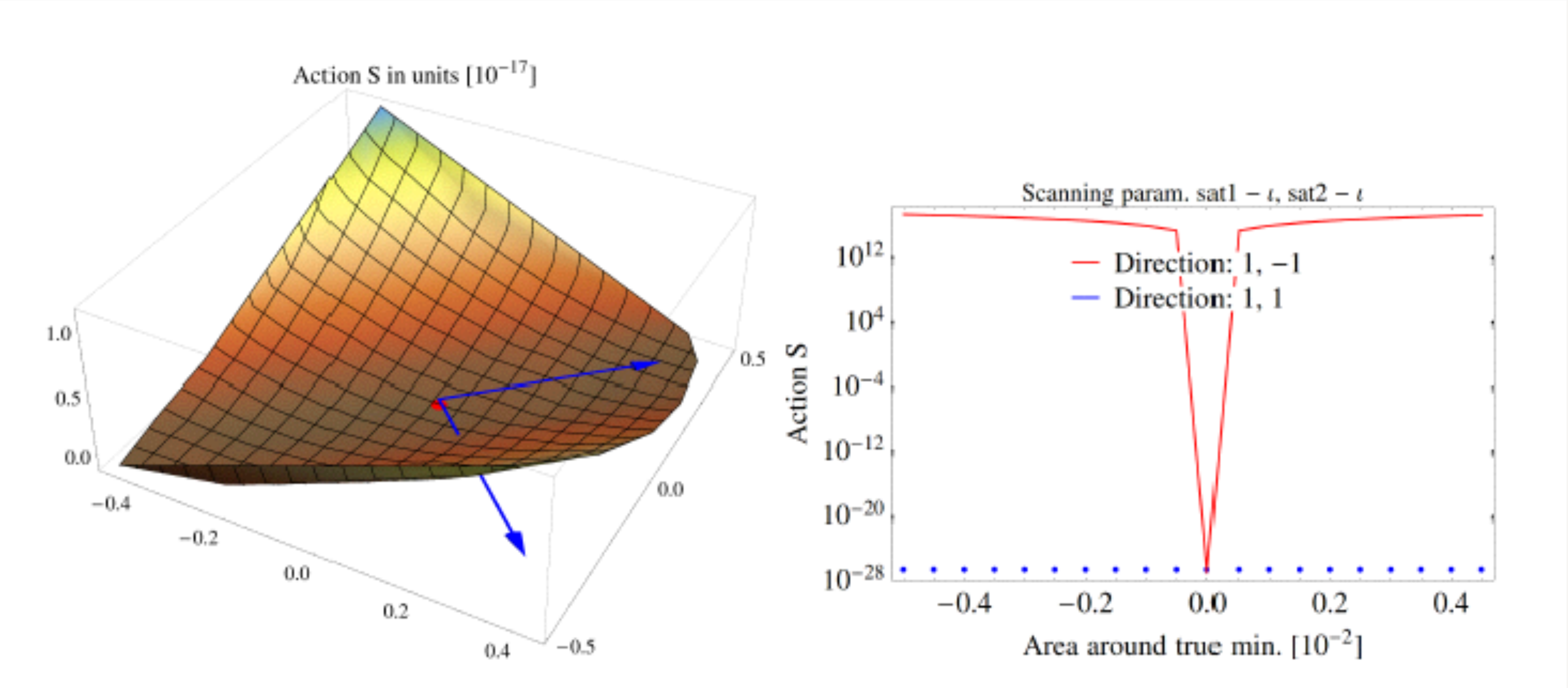
DETERMINATION OF ORBITAL PARAMETERS



DEGENERACIES

- between orbital parameters
- Hessian matrix



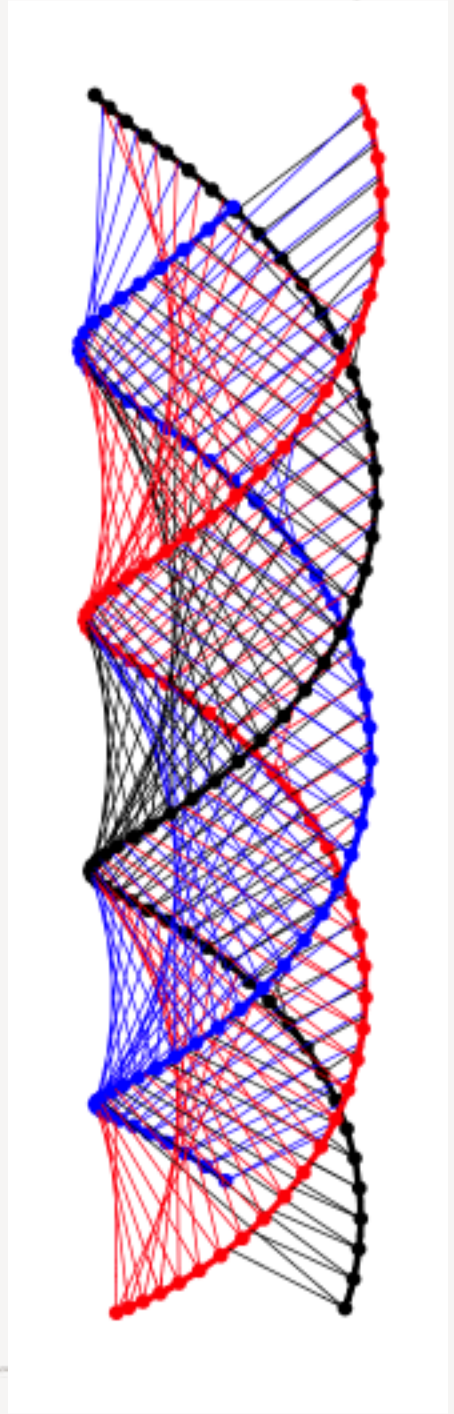


- degeneracy in: ι_1-ι_2, Ω_1-Ω_2, t_{a1}-t_{a2}

ABC CONCEPT - ADVANTAGES

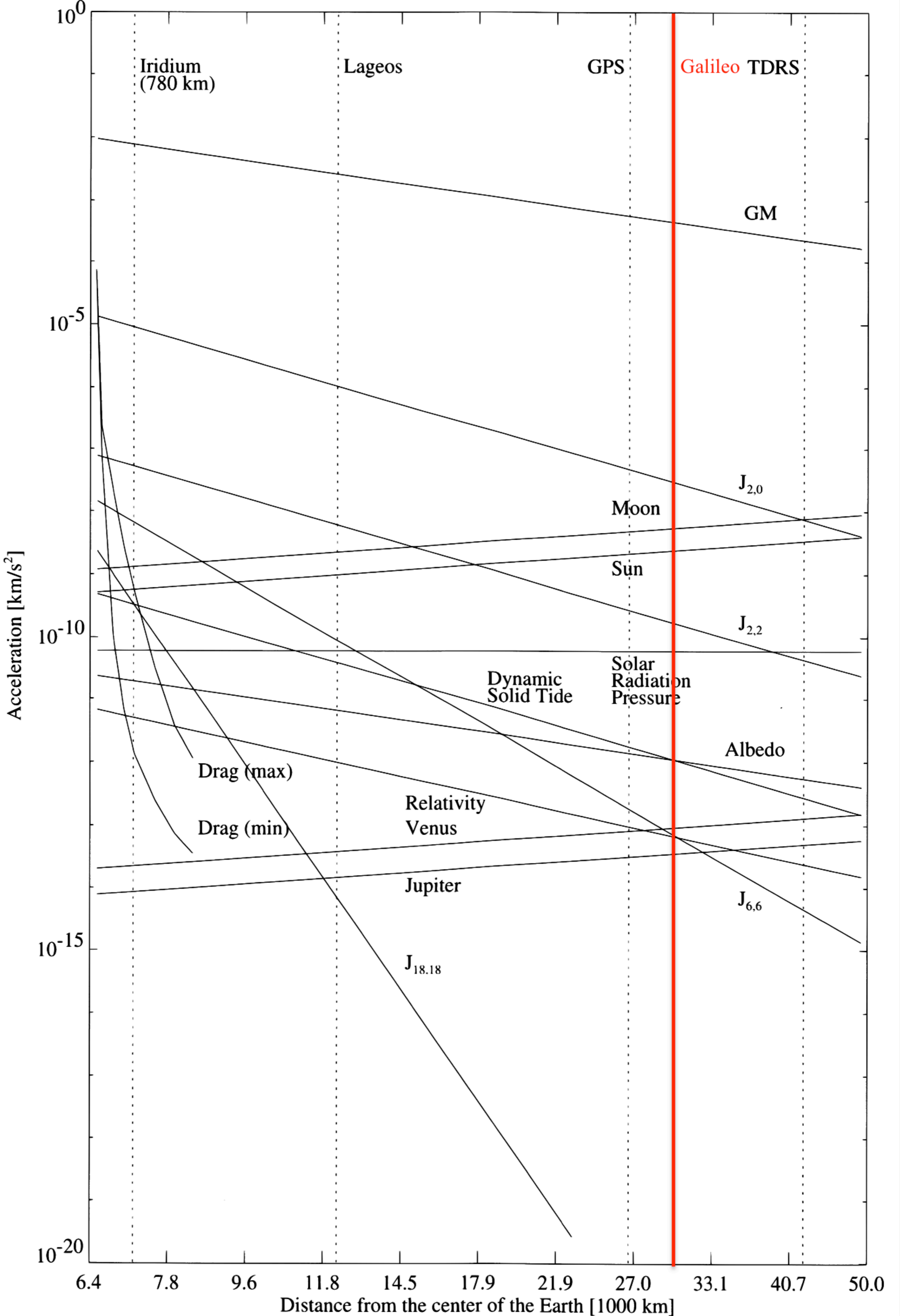
- its realisation does not depend on observations from Earth
  - no entanglement with Earth internal dynamics, no Earth stations for maintaining of reference frame
- robustness of recovering orbital parameters with respect to noise in the data
- consistency of description with redundant number of satellites
- stability and accuracy
  - based on well-known satellite dynamics, satellite orbits are very stable in time, and can be accurately described
- applications in science

geophysics, relativistic gravitation and reference frames, determine/refine values of gravitational parameters (e.g. multipoles)…



ESA PECS RGNSS PROJECT 2011-2014

- more realistic case: non-spherically symmetric Earth + other celestial bodies
- all gravitational perturbations: Earth multipoles, tides, Moon, Sun, Jupiter Venus, Kerr effect (due to Earth rotation)
- do positioning and ABC concept still work?



Montenbruck and Gill, 2005

1. INCLUDING GRAVITATIONAL PERTURBATIONS IN SPACETIME METRIC

- Schwarzschild background (spherically symmetric, time independent) + linear perturbations

perturbed metric: $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} + O(h^2)$ $\quad (h_{\mu\nu} \ll g^{(0)}_{\mu\nu})$

$$g_{\mu\nu} = \underbrace{g^{(0)}_{\mu\nu}}_{\text{Schwarzschild}} + \underbrace{h^{(2-6)}_{\mu\nu} + h^{(planets)}_{\mu\nu} + h^{(tides)}_{\mu\nu}}_{\text{Regge-Wheeler-Zerilli multipole expansion}} + h^{(Kerr)}_{\mu\nu}$$

Schwarzschild

Regge-Wheeler-Zerilli multipole expansion

IN PERTURBED METRIC

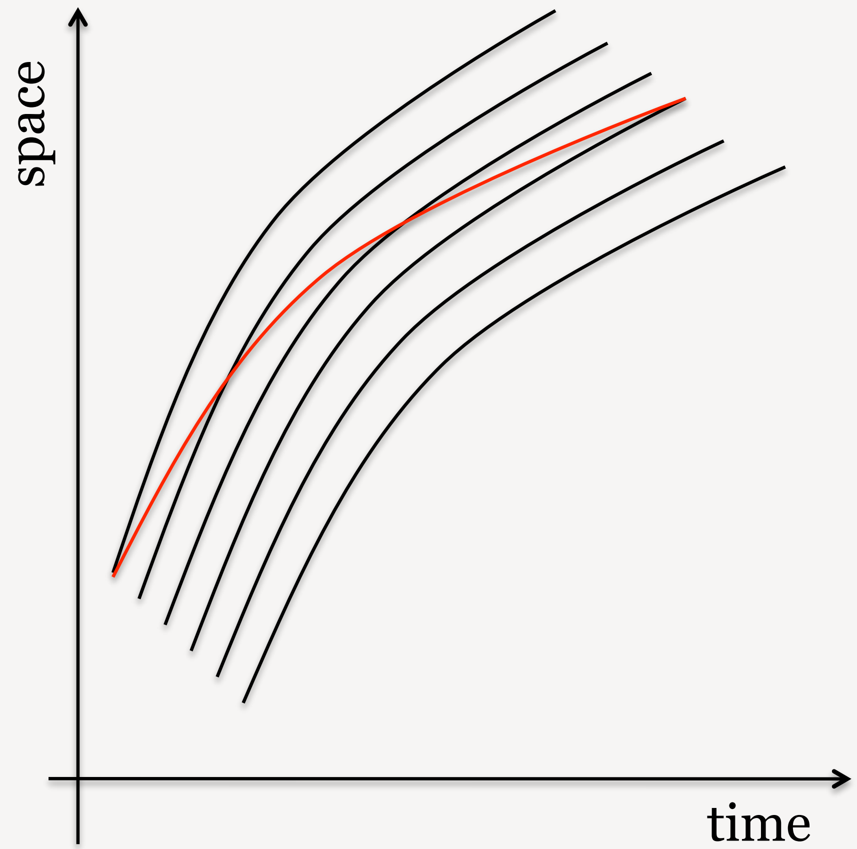
- not constants of motion anymore: $H, \; a, \; \varepsilon, \; \iota, \; -\tau_a, \; -t_a, \; \omega, \; \Omega$

- satellite dynamics: perturbed satellite orbits (slow time evolution of orbital parameters)

using

- geodesics approach:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$



or

- Hamiltonian formalism

$$H = \frac{1}{2} g^{(0)\mu\nu} p_\mu p_\nu - \frac{1}{2} h^{\mu\nu} p_\mu p_\nu = H^{(0)} - \Delta H$$

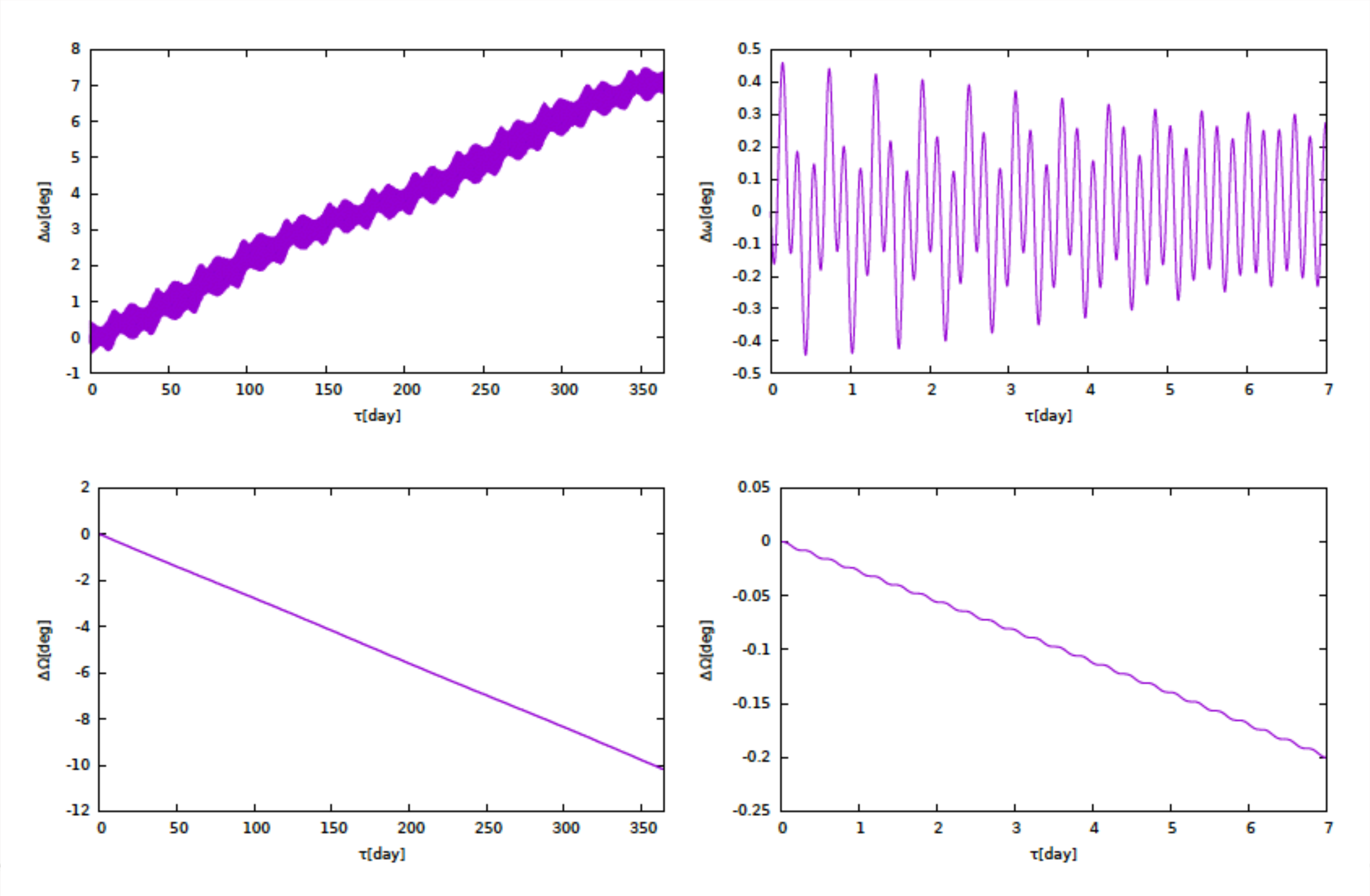
- evolution of orbital parameters

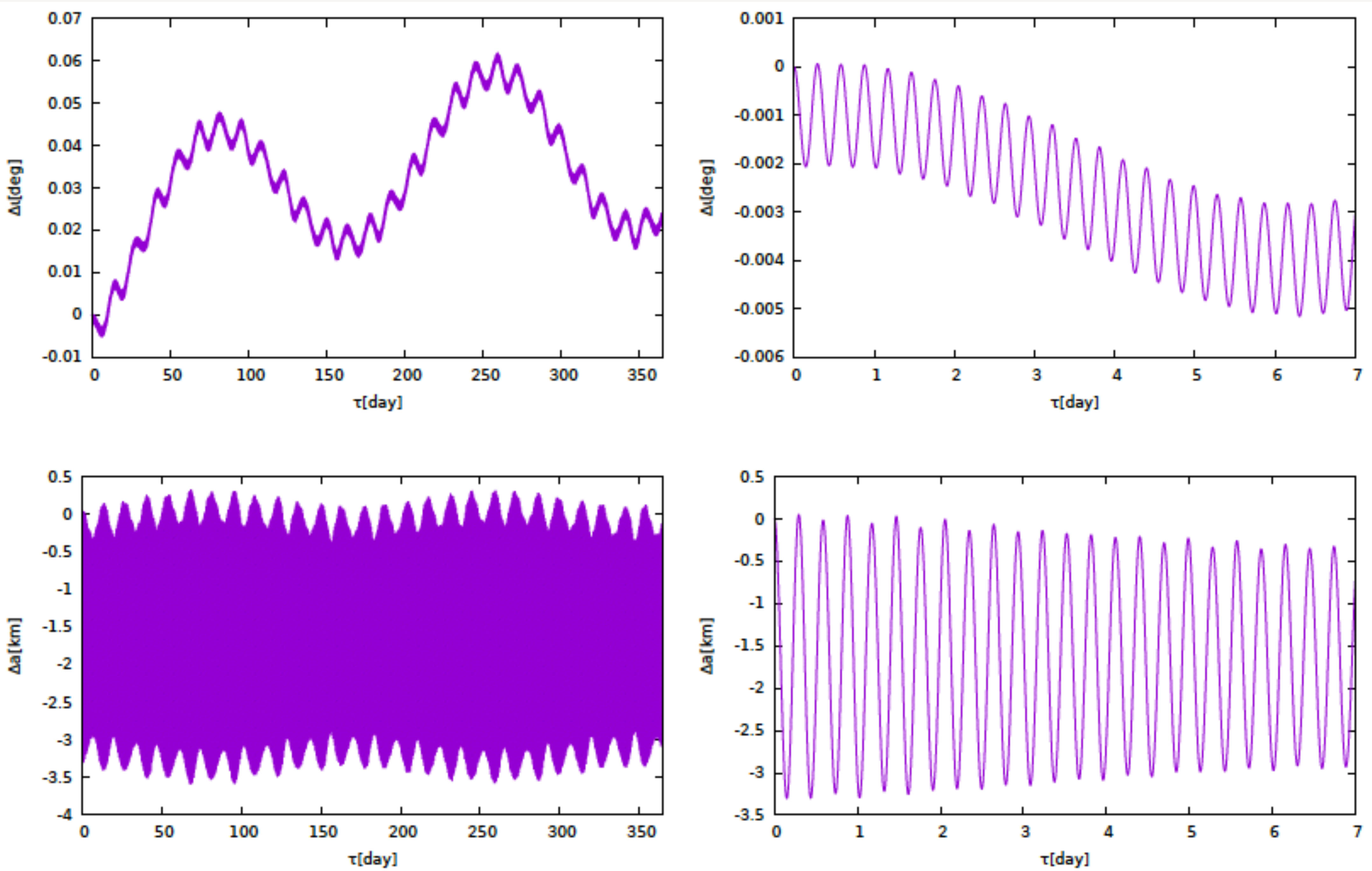
$$\dot{Q}^k = \left.\frac{\partial H}{\partial P_k}\right|_{Q^k, P_k} = -\left.\frac{\partial \Delta H}{\partial P_k}\right|_{Q^k, P_k} = -\frac{1}{2} \frac{\partial (h^{\mu\nu} p_\mu p_\nu)}{\partial P_k}$$

$$\dot{P}_k = -\left.\frac{\partial H}{\partial Q^k}\right|_{Q^k, P_k} = \left.\frac{\partial \Delta H}{\partial Q^k}\right|_{Q^k, P_k} = \frac{1}{2} \frac{\partial (h^{\mu\nu} p_\mu p_\nu)}{\partial Q^k} .$$

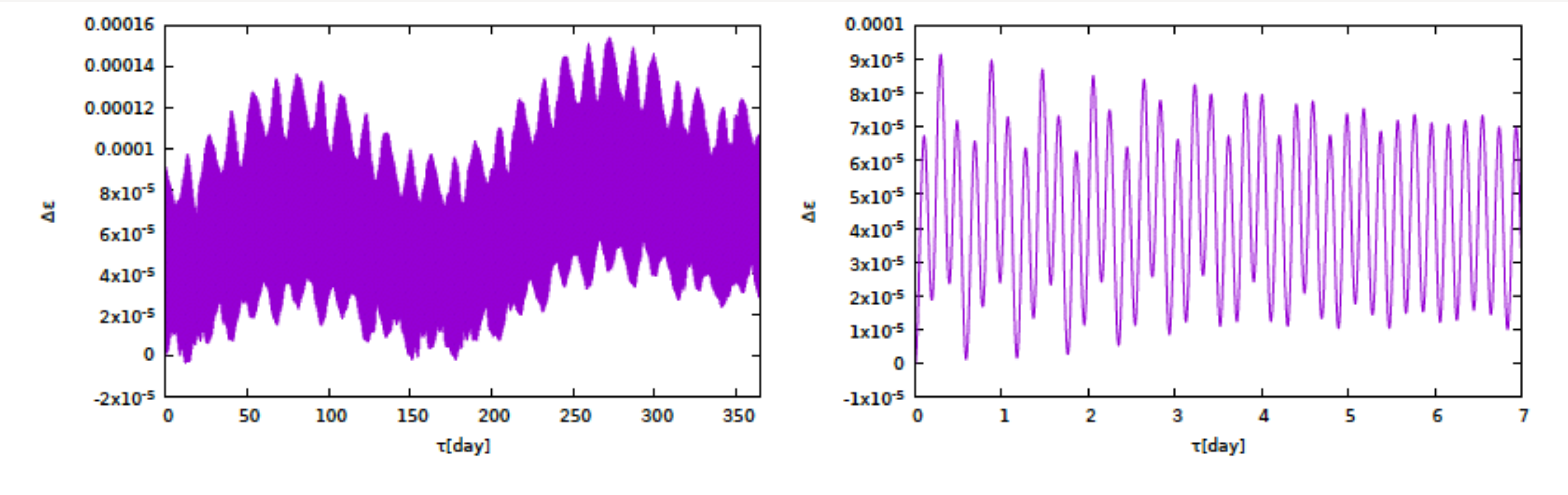
- use of analytical solutions for orbits (Kostić 2012)

2. PERTURBED ORBITS - EVOLUTION OF ORBITAL PARAMETERS





Δι[deg]
τ[day]
Δι[deg]
τ[day]
Δa[km]
τ[day]
Δa[km]
τ[day]



0.00016
0.00014
0.00012
0.0001
8x10-5
6x10-5
4x10-5
2x10-5
0
-2x10-5
Δε
0
50
100
150
200
250
300
350
τ[day]
0.0001
9x10-5
8x10-5
7x10-5
6x10-5
5x10-5
4x10-5
3x10-5
2x10-5
1x10-5
0
-1x10-5
Δε
0
1
2
3
4
5
6
7
τ[day]

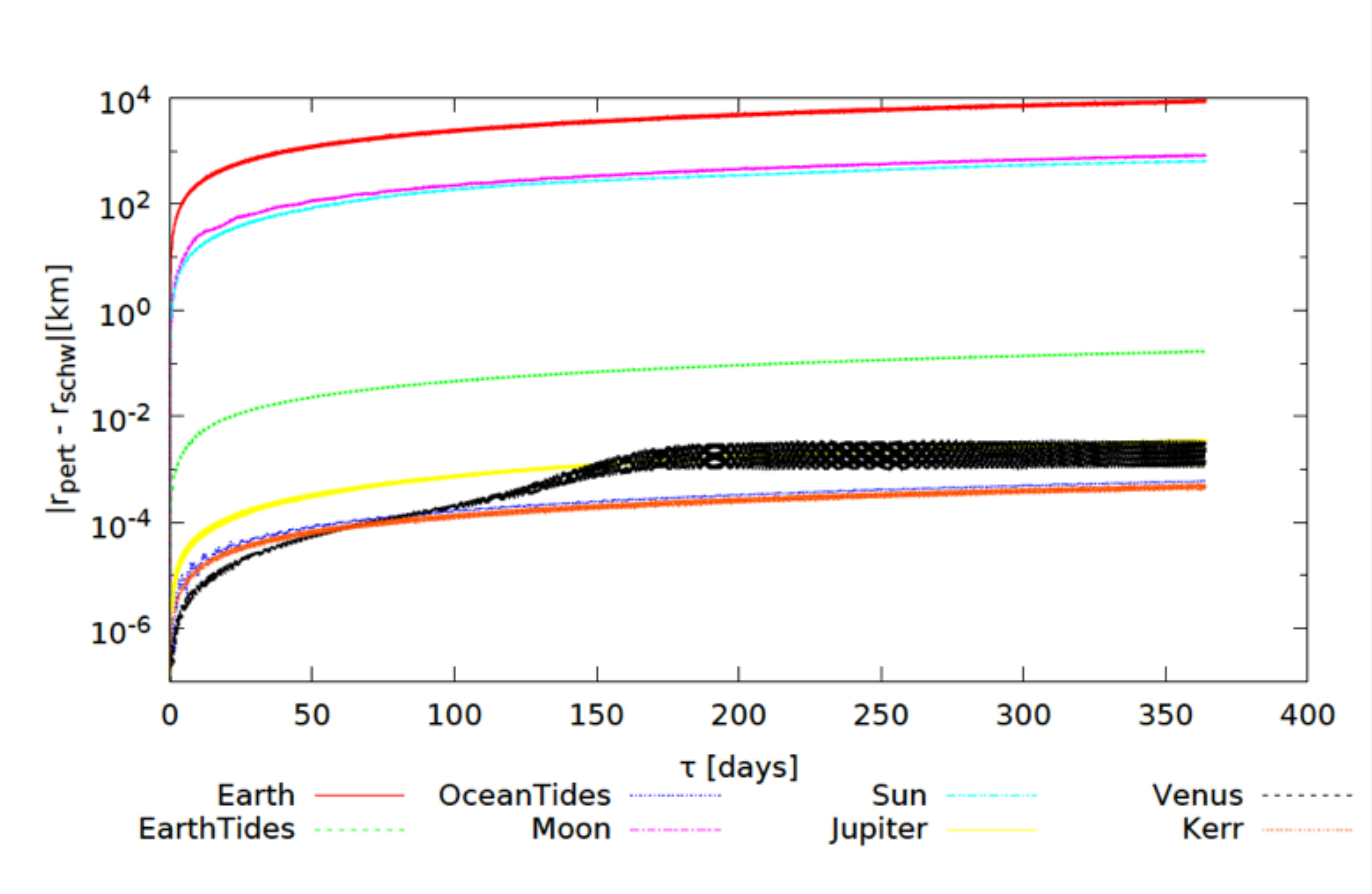
The amplitudes of oscillations of orbital parameters due to perturbations.

perturbation	$\Delta\omega$	$\Delta\Omega$	$\Delta\iota$	Δa	$\Delta\varepsilon$
Earth multipoles	$0.4°$	$18''$	$4''$	1.5 km	4×10^{-5}
solid tide	$0.05''$	$2 \times 10^{-4''}$	$10^{-4''}$	5 cm	4×10^{-9}
ocean tide	$10^{-3''}$	$3 \times 10^{-6''}$	$3 \times 10^{-6''}$	0.6 mm	3×10^{-11}
Moon	$0.8°$	$1''$	$0.7''$	300 m	10^{-5}
Sun	$2'$	$0.7''$	$0.2''$	100 m	5×10^{-6}
Venus	$5 \times 10^{-5''}$	$4 \times 10^{-7''}$	$4 \times 10^{-7''}$	0.2 mm	10^{-11}
Jupiter	$2 \times 10^{-3''}$	$2 \times 10^{-6''}$	$3 \times 10^{-6''}$	1 mm	5×10^{-11}
Kerr	$3.6 \times 10^{-5''}$	$10^{-6''}$	$3 \times 10^{-6''}$	5×10^{-14} m	2×10^{-12}

Table 2: The secular contribution of perturbations to evolution of orbital parameters. The values are per year.

perturbation	$(\mathrm{d}\omega/\mathrm{d}t)_{\mathrm{sec}}$	$(\mathrm{d}\Omega/\mathrm{d}t)_{\mathrm{sec}}$	$(\mathrm{d}\iota/\mathrm{d}t)_{\mathrm{sec}}$	$(\mathrm{d}a/\mathrm{d}t)_{\mathrm{sec}}$	$(\mathrm{d}\varepsilon/\mathrm{d}t)_{\mathrm{sec}}$
Earth multipoles	5°	-9.5°	0	0	2×10^{-5}
solid tide	$0.2''$	$-0.3''$	0	0	0
ocean tide	0	0	0	0	0
Moon	1.75°	-0.55°	$3'$	0	2×10^{-5}
Sun	0.7°	-0.25°	0	0	0
Venus	$0.05''$	$0.025''$	$6\times10^{-3}{}''$	0	1.6×10^{-9}
Jupiter	$0.007''$	$-0.01''$	$-0.005''$	0	1.2×10^{-9}
Kerr	$-0.004''$	$2.5\times10^{-3}{}''$	0	0	0

EFFECT OF ALL GRAVITATIONAL PERTURBATIONS ON POSITION



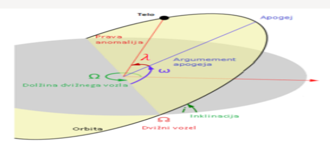
3. POSITIONING IN PERTURBED SPACETIME

- model positioning: receiver + signals from 4 satellites on perturbed orbits
- exchange of signals: ray-tracing in Schwarzschild metric
- proper times $\longrightarrow$ receiver's position
- positioning - works!
- relative accuracy 10^{-32} - 10^{-30} in t, 10^{-28} - 10^{-26} in x,y,z (!)
- on a laptop: in 0.04 s (no "last position" known)

ABC SYSTEM

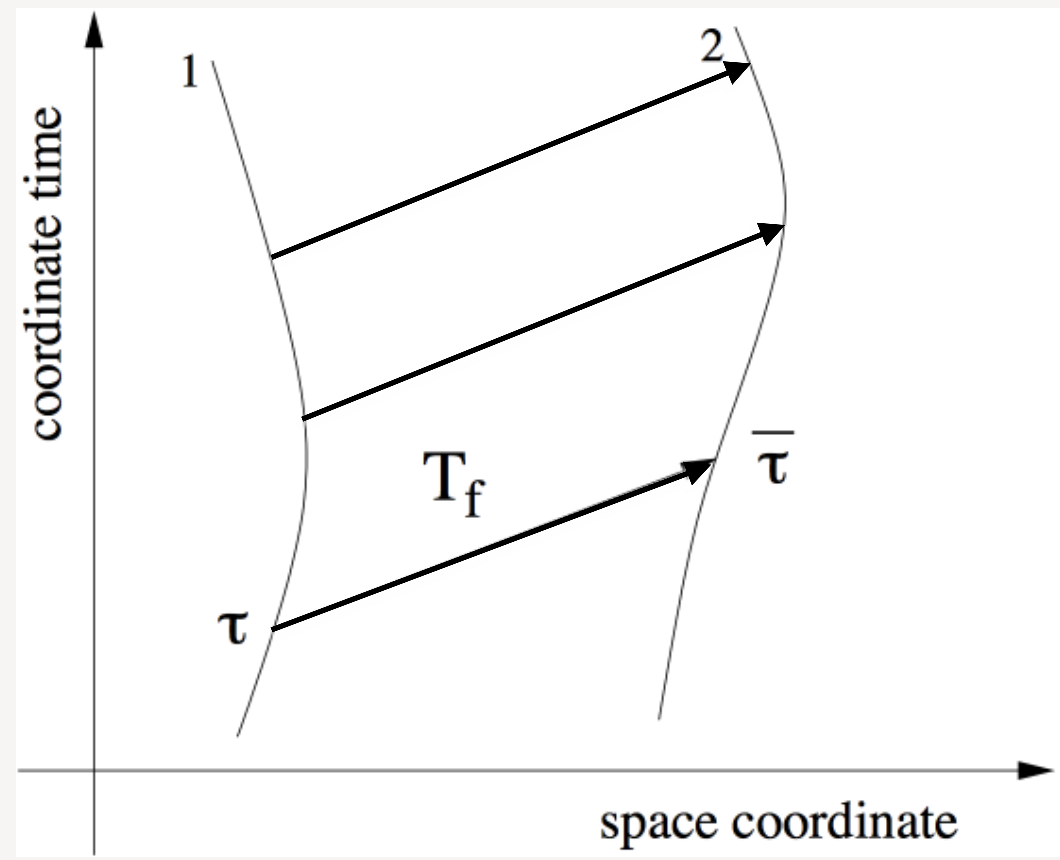
- initially, orbital parameters known only with limited accuracy, use of inter-satellite links (pairs τ_1, τ_2)
- time of flight:

$$g_{\mu\nu} = \begin{bmatrix} -(1-\frac{2GM}{rc^2}) & 0 & 0 & 0 \\ 0 & \frac{1}{1-2GM/rc^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$



- action S:

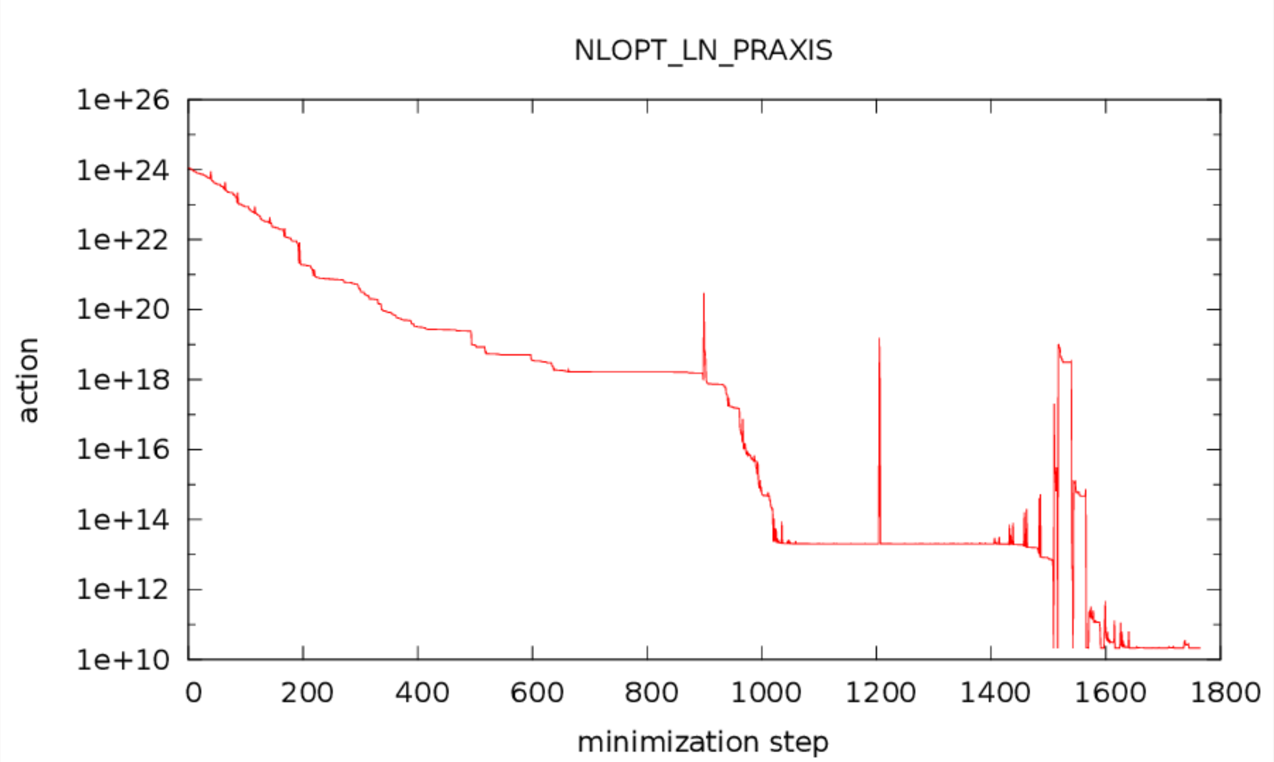
$$S\left(Q^\mu(0), P_\mu(0)\right) = \sum_k (t_2(\overline{\tau}) - t_1(\tau) - T_f(\vec{R}_1(\tau), \vec{R}_2(\overline{\tau}))^2$$

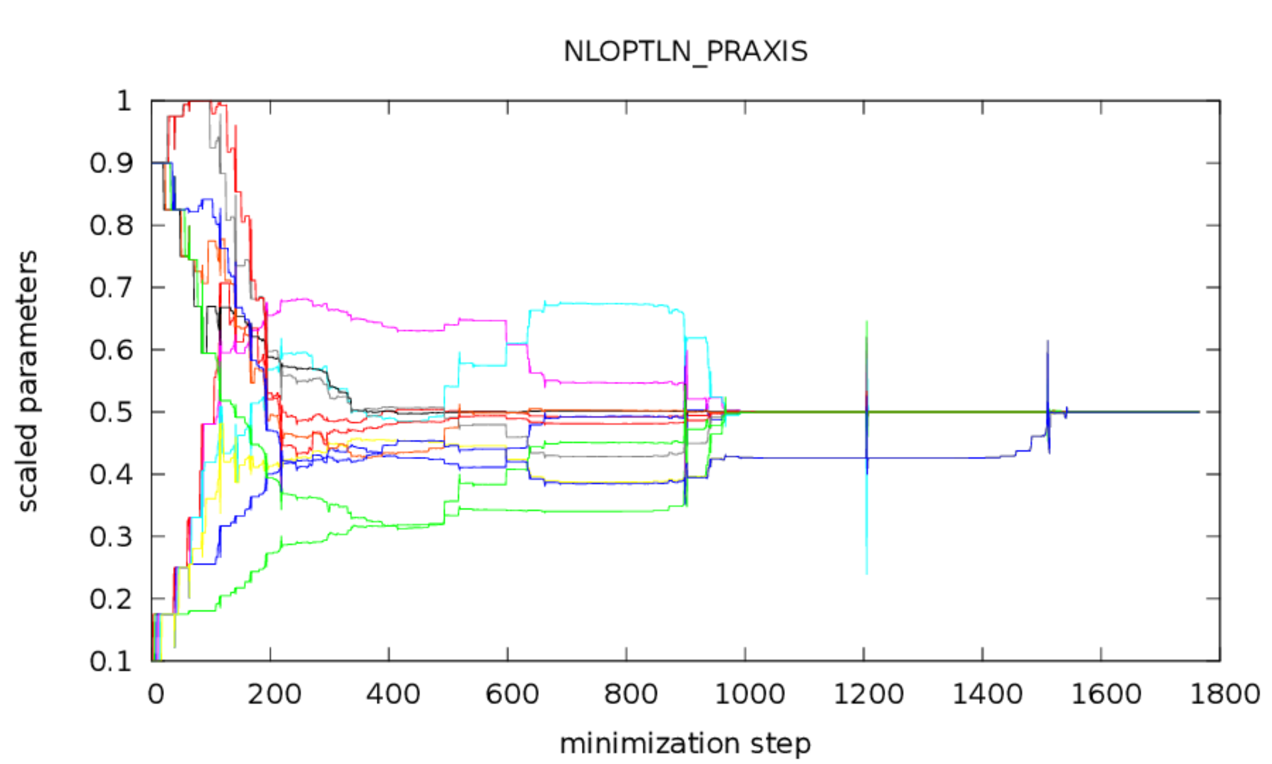


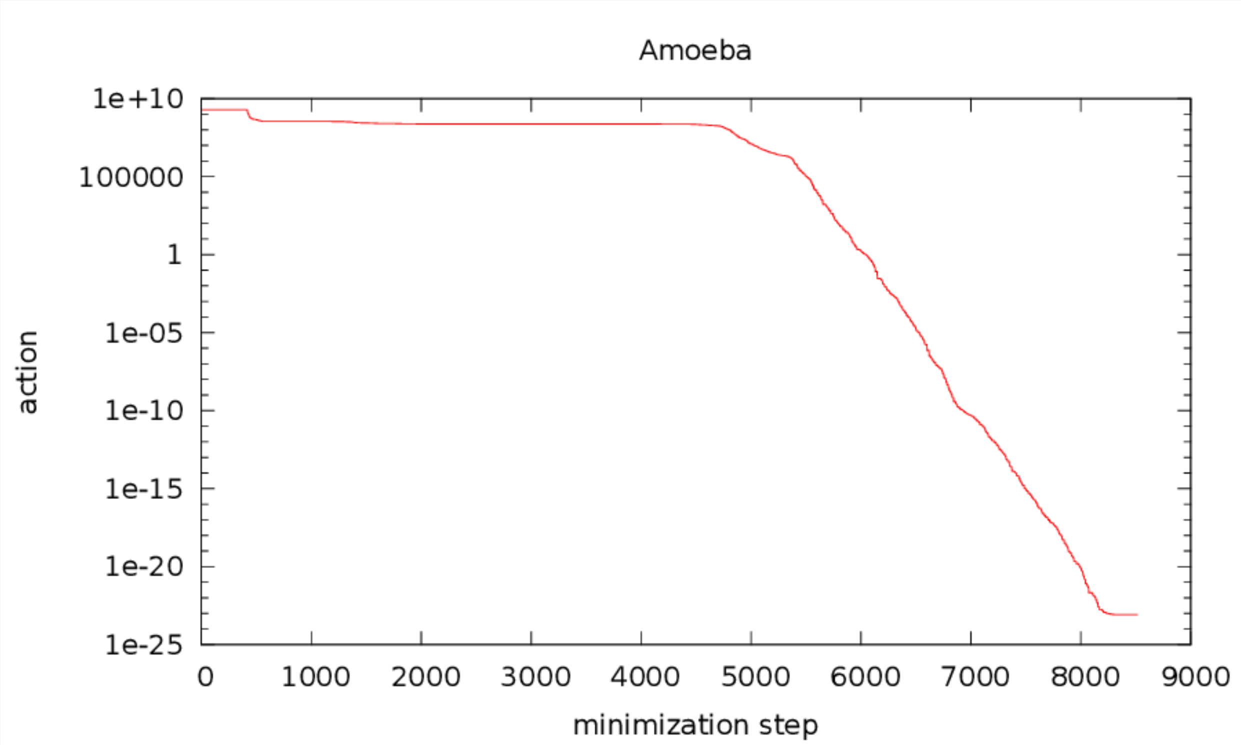
- minimization of S
- possible to refine orbital parameters to relative accuracy 10^{-22} - works!

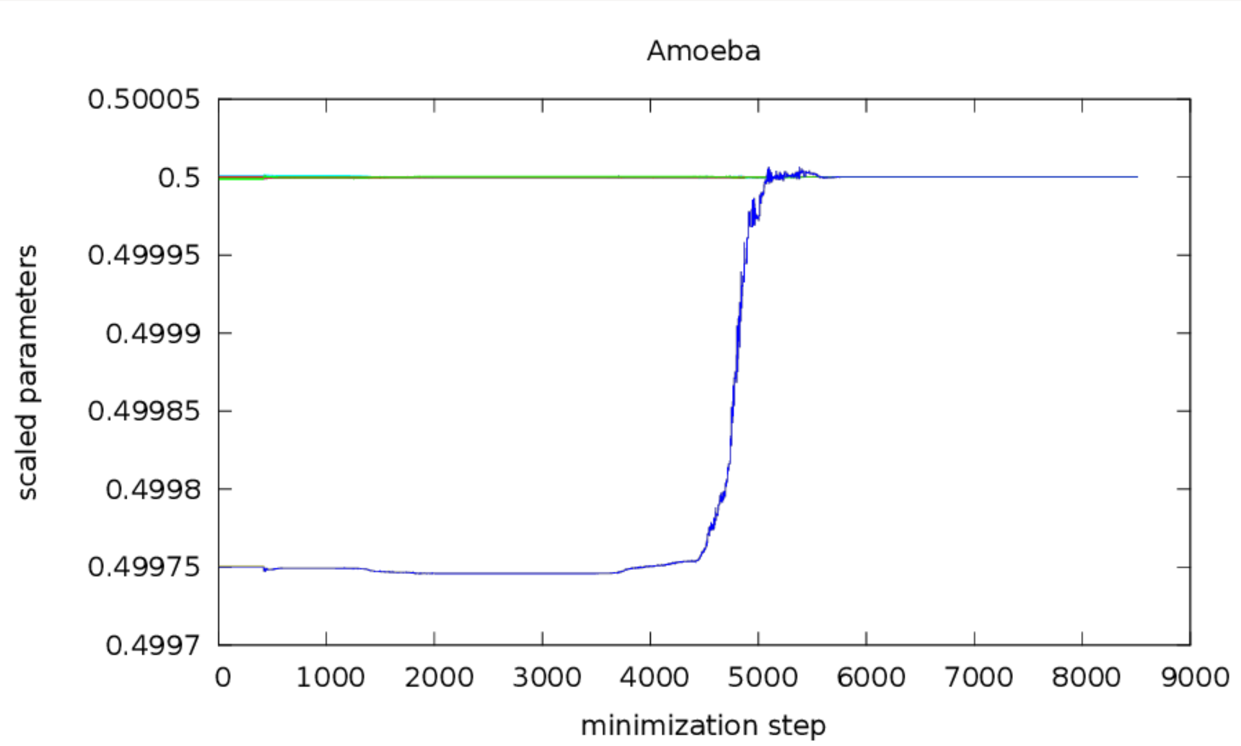
MINIMIZATION…

12D minimization



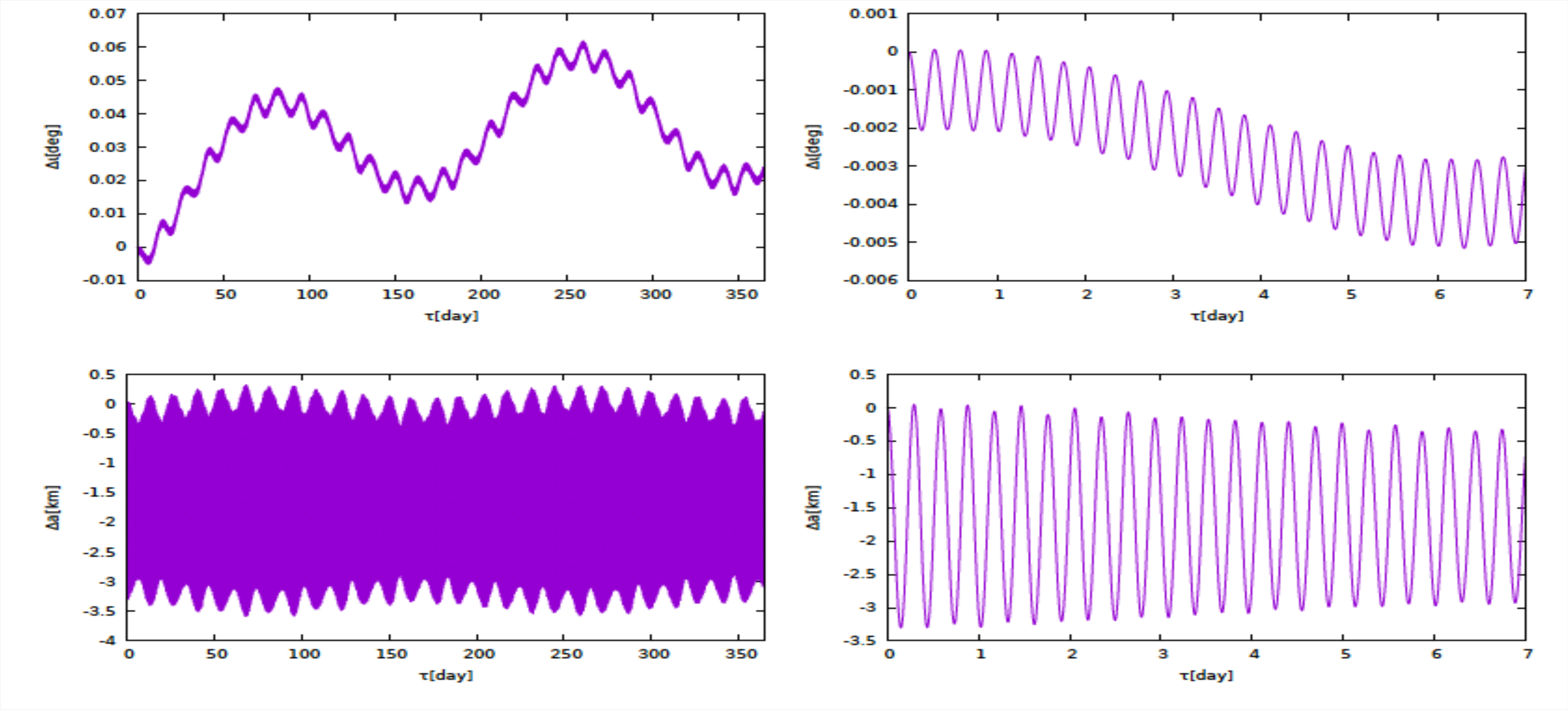



Amoeba
action
1e+10
100000
1
1e-05
1e-10
1e-15
1e-20
1e-25
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
minimization step


Amoeba
scaled parameters
0.50005
0.5
0.49995
0.4999
0.49985
0.4998
0.49975
0.4997
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
minimization step

NO DEGENERACIES

- Schwarzschild space-time + Earth multipoles + solid tides + ocean tides

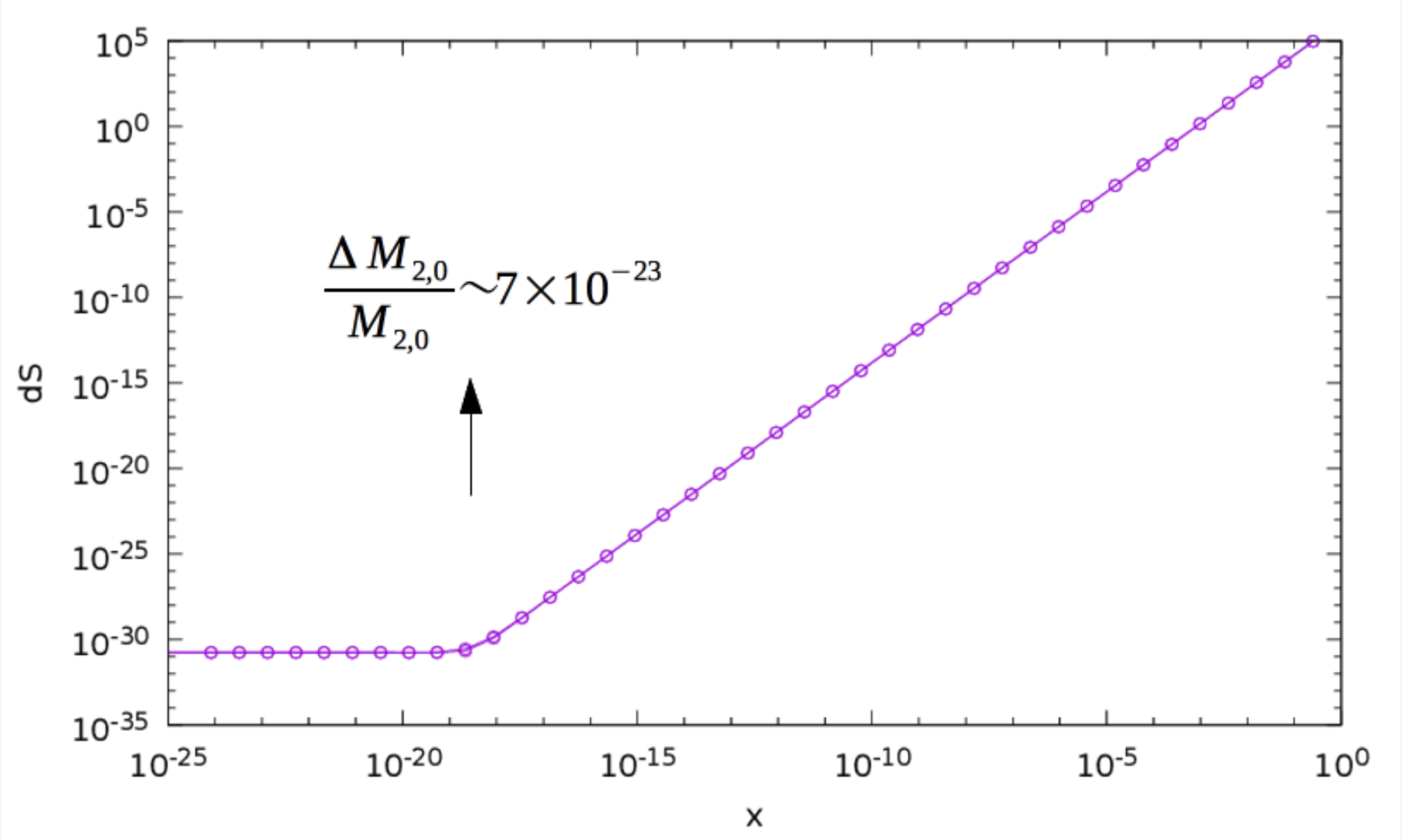


PROBE THE SPACETIME

- redundant satellites - increase accuracy and stability
- with more than 4 satellites we can probe the spacetime!
- infer spacetime metric;
- geo-sciences: interior structure of the Earth, ocean currents, continential drift…

4. REFINEMENT OF GRAVITATIONAL PARAMETERS

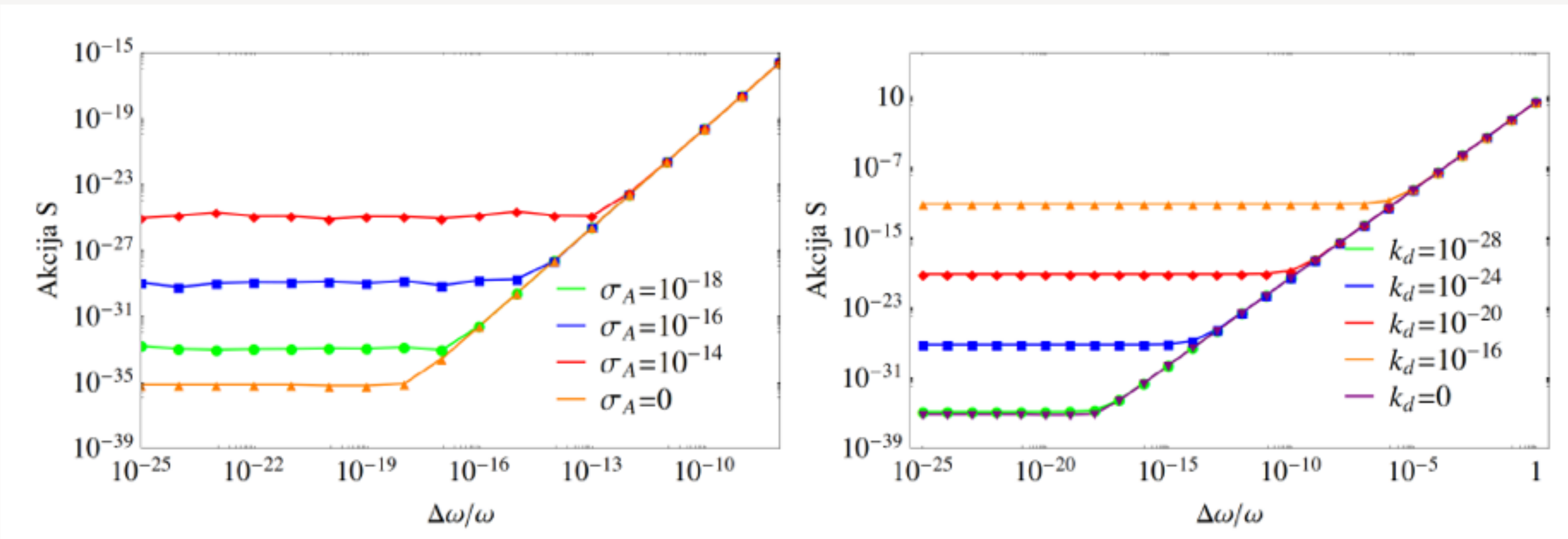
- is it possible to refine values of gravitational parameters?
- *theoretically* yes, because there is a well defined minimum in the action S



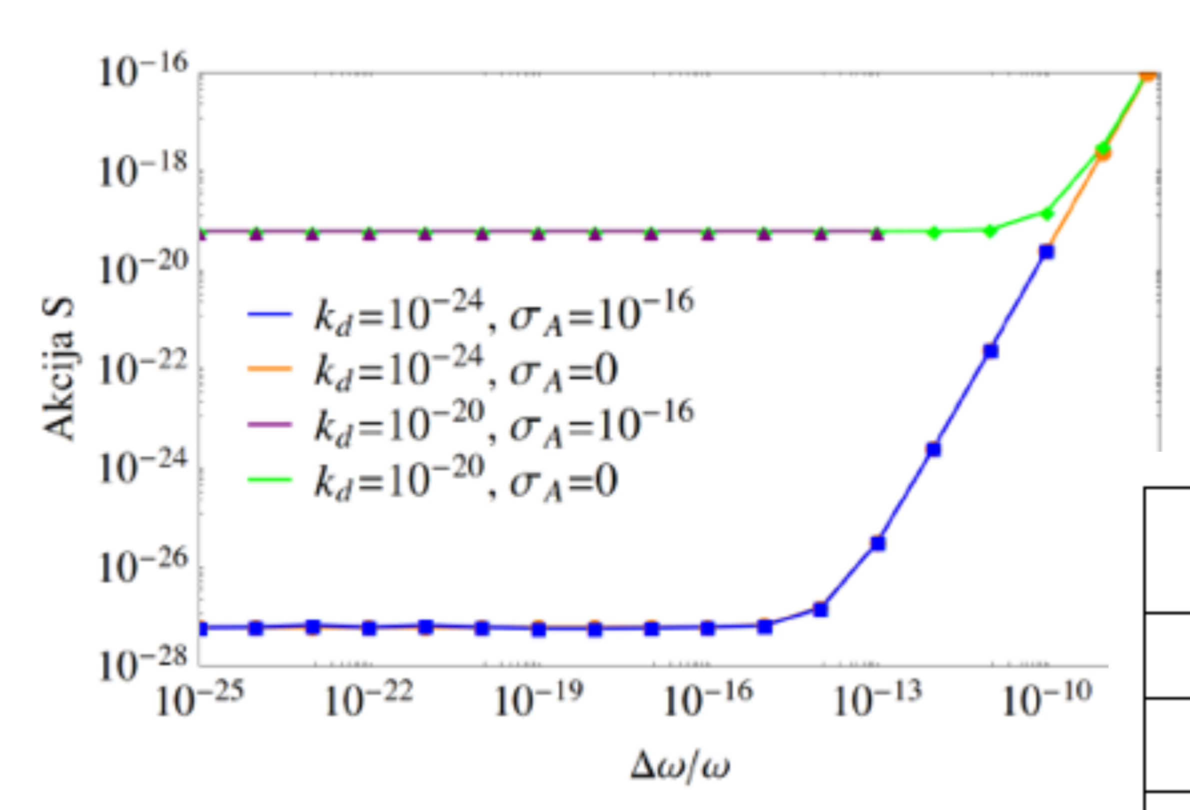
parameter P	$\frac{\Delta P}{P}$	$S\ [(\frac{r_g}{c})^2]$	ΔL [m]	$(\frac{\Delta P}{P})_{\text{knee}}$
$\Omega_\oplus$	$1.4 \cdot 10^{-8}$	$1.1 \cdot 10^{-6}$	0.00048	10^{-21}
$M_{2,0}$	$7 \cdot 10^{-8}$	1.5	0.1	$7 \cdot 10^{-23}$
Re $M_{2,1}$	$5 \cdot 10^{-21}$	$1 \cdot 10^{-31}$	$8 \cdot 10^{-24}$	$> 5 \cdot 10^{-18}$
Im $M_{2,1}$	$8 \cdot 10^{-22}$	$1 \cdot 10^{-31}$	$4 \cdot 10^{-21}$	$> 8 \cdot 10^{-19}$
Re $M22$	0.00002	10	0.38	$2 \cdot 10^{-20}$
Im $M22$	0.00004	12	0.002	$4 \cdot 10^{-20}$
$M_☾$	0.001	$4.6 \cdot 10^{6}$	140	10^{-21}
$r_☾$	0.001	$2 \cdot 10^{7}$	261	10^{-21}
$M_\odot$	0.001	71000	113	10^{-21}
$r_\odot$	0.001	$2.8 \cdot 10^{6}$	220	10^{-21}
$M_♀$	0.001	$4.2 \cdot 10^{-7}$	0.00008	10^{-14}
$r_♀$	0.001	$1.5 \cdot 10^{-6}$	0.00016	10^{-15}
$M_♃$	0.001	0.000086	0.00046	10^{-14}
$r_♃$	0.001	0.0003	0.00084	10^{-16}

CLOCK NOISE

- Allan deviation: $\sigma_A(\tau) = \sqrt{\frac{1}{2}\langle(\nu_{n+1} - \nu_n)^2\rangle}$
- drift: $\nu_m = \nu_0(1 + k_d)$



- omega = Earth angular velocity



Parameter P	current $\delta P = \frac{\Delta P}{P}$	$\delta P = \left(\frac{\Delta P}{P}\right)_{knee}$
$\omega_{\oplus}$	$1.4 \cdot 10^{-8}$	10^{-17}
$M_{2,0}$	$7.0 \cdot 10^{-8}$	10^{-11}
$M_{☾}$	$1.0 \cdot 10^{-9}$	10^{-12}
$r_{☾}$	$6.4 \cdot 10^{-11}$	10^{-12}
$M_{\odot}$	$1.2 \cdot 10^{-8}$	10^{-11}
$r_{\odot}$	$6.6 \cdot 10^{-9}$	10^{-11}
$L_{\odot}$	$3.6 \cdot 10^{-5}$	10^{-9}

$(\sigma_A = 10^{-14}\ \mathrm{s}^{-1},\ k_d = 10^{-24})$

SOLAR RADIATION EFFECT

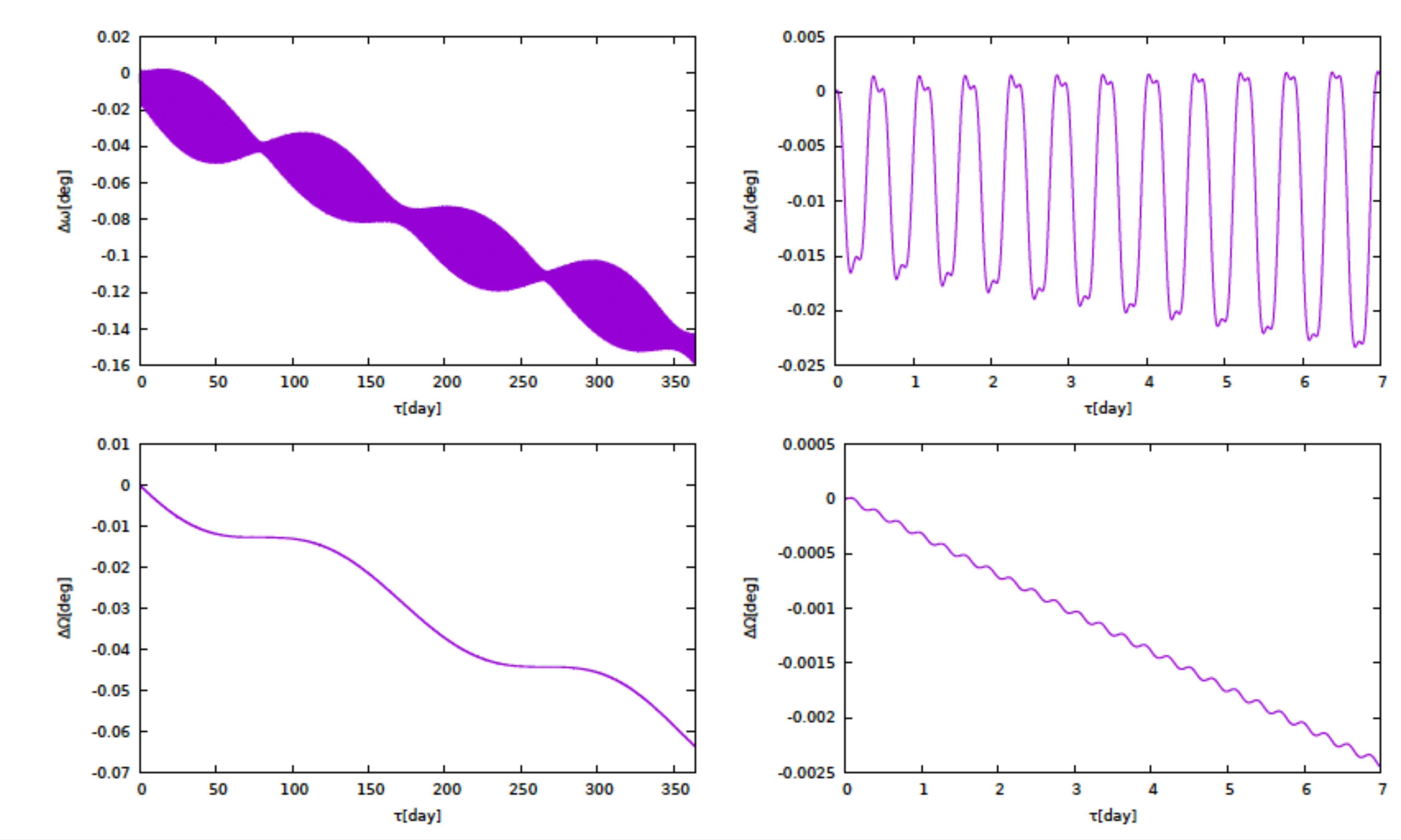
- simple Solar radiation model

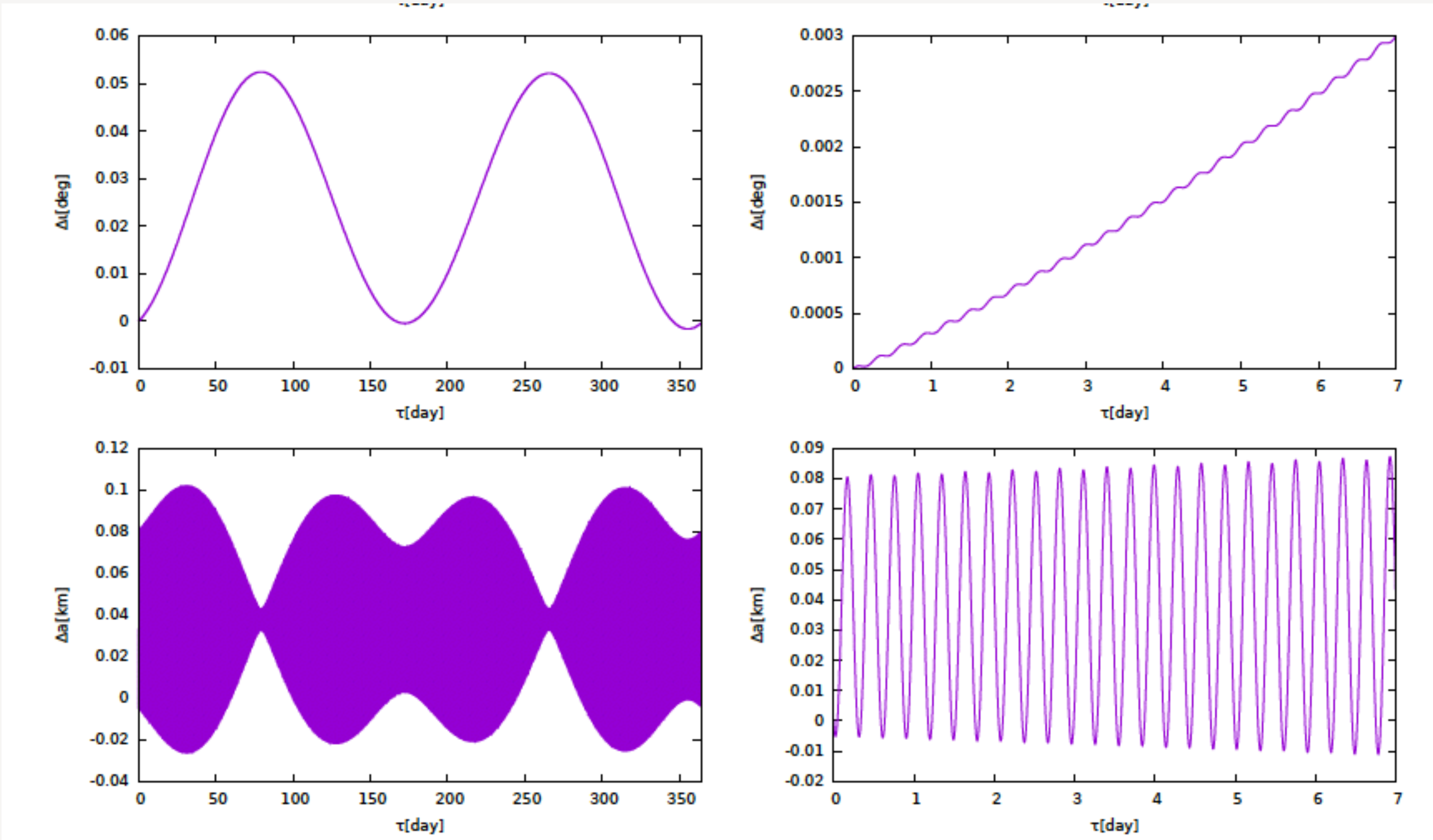
(no eclipses, panels perepndicular to r_{ss}): $\mathbf{F_{st}} = -I_{\odot} C_R A \frac{\mathbf{r_{ss}}}{r_{ss}^3}$

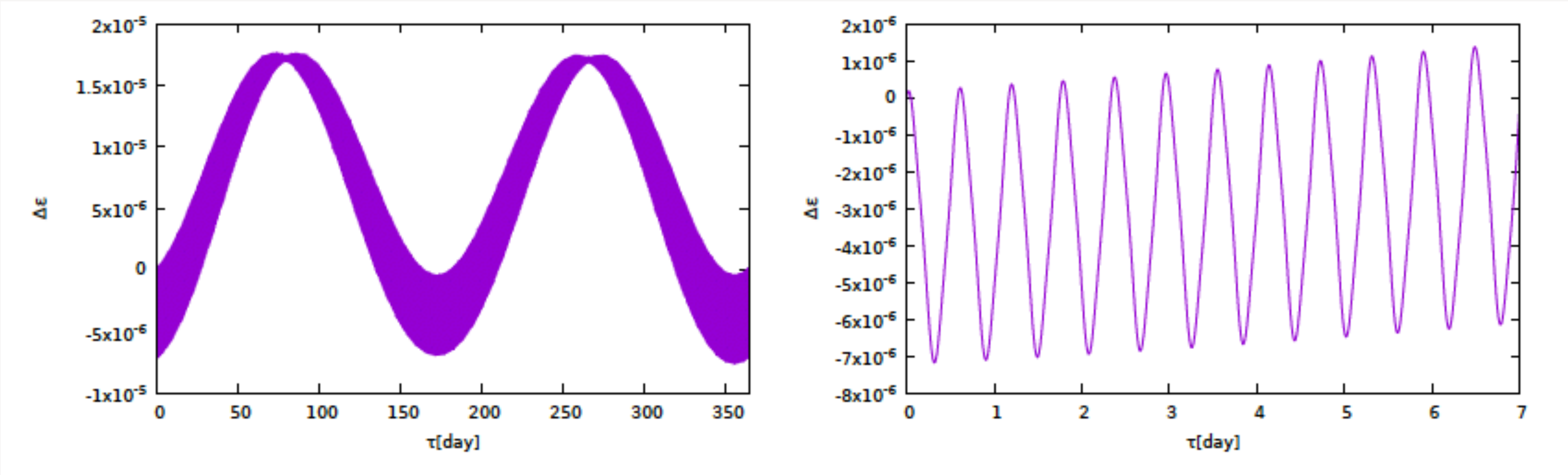
- potential: $\mathbf{F_{st}} = -\nabla \phi_{st}$
- in the metric:

$$h_{\mu\nu}^{SS} = \begin{bmatrix} -\frac{2C}{r_{ss}} & 0 & 0 & 0 \\ 0 & -\frac{2C}{r_{ss}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ORBITAL PARAMETERS EVOLUTION




Δι[deg]
τ[day]
Δι[deg]
τ[day]
Δa[km]
τ[day]
Δa[km]
τ[day]



2x10-5
1.5x10-5
1x10-5
5x10-6
0
-5x10-6
-1x10-5
Δε
0
50
100
150
200
250
300
350
τ[day]
2x10-6
1x10-6
0
-1x10-6
-2x10-6
-3x10-6
-4x10-6
-5x10-6
-6x10-6
-7x10-6
-8x10-6
Δε
0
1
2
3
4
5
6
7
τ[day]

POSSIBLE PROBLEMS IN PRACTICE IN AUTONOMOUS RPS

- non-gravitational perturbations, clock (hardware) noise…
- effectiveness of numerical multi-dimensional minimization methods
- even more severe minimization problems for all orbital and gravitational parameters at the same time
- …

SUMMARY OF RESULTS

- relativistic positioning and ABC system in the realistic gravitational field (with gravitational perturbations) are (numerically) feasible, accurate and stable
- theoretically possible to “measure” the gravitational field of the Earth and nearby celestial bodies - independent way to measure space-time in the vicinity of Earth - various scientific applications

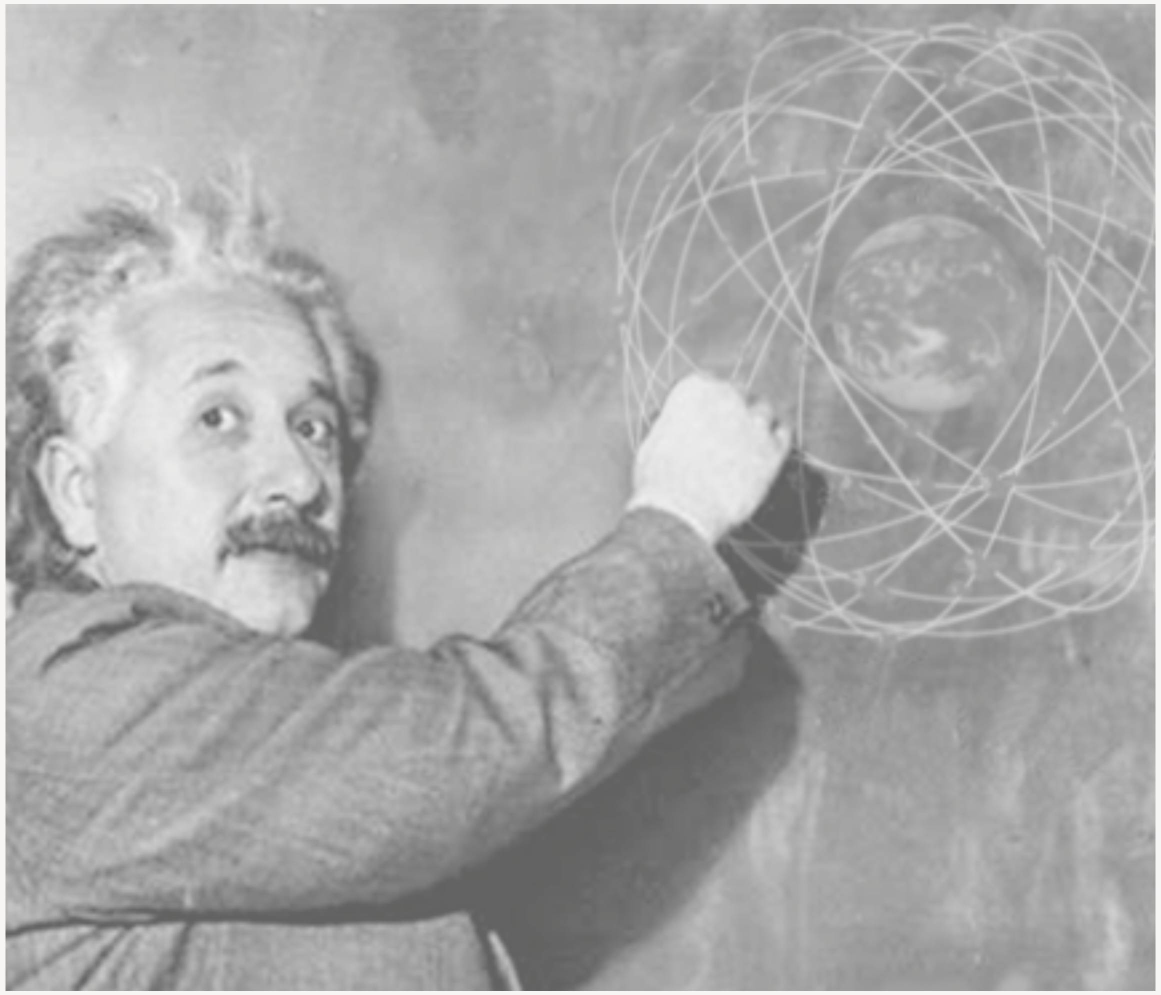
POSSIBLE NEXT STEPS

- study of suitable methods for highly accurate and faster minimisation
- study of influence of non-gravitational perturbations on relativistic positioning and ABC
- feasibility study on ground-based infrastructure and on-board hardware required for implementing ABC (put receivers on 2 Galileo satellites?)
- feasibility study of a system of small satellites with inter-satellite links for ABC concept demonstration
- …

TEST REQUIREMENTS

- 2 satellites
- with accurate clocks and inter-satellite links
- store pairs (τ_1, τ_2) along min. 1-2 orbits
- download them
- minimization and orbits determination on the ground
- comparisson with orbits determined via other methods (GPS, ground tracking…)

ADVANTAGES OF RELATIVISTIC + ABC CONCEPT



- accuracy, stability and lower costs of such a system - no ground tracking (link between a terrestrial reference frame and ABC established by several receivers at known terrestrial positions)
- no clock synchronisation
- independent, robust, consistent
- very promising!

THANK YOU!